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MATHEMATICAL THEORY

OF

ELECTRICITY AND MAGNETISM

EMTAGE

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An Introduction to the Mathematical Theory of Electricity and Magnetism

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PREFACE TO THE SECOND EDITION

IN this edition I have corrected some mistakes, and made some additions that seemed essential.

I trust that some degree of originality (which I have not hitherto claimed) will be recognized in the treatment of certain subjects.

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PREFACE TO THE FIRST EDITION

THIS book has been written with the object of supplying an Introduction to the Mathematical treatment of Electricity and Magnetism, and will, it is hoped, be useful to those who possess the requisite elementary knowledge of Differential and Integral Calculus. It is complete in itself, and may be read without previous knowledge of the subject. Purely experimental parts of the subject, requiring no special mathematical treatment, have been entirely omitted.

The Author is especially indebted for assistance to the Treatises of Professor Clerk Maxwell, and of Messrs. Mascart and Joubert.

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AN INTRODUCTION

TO

THE MATHEMATICAL THEORY OF ELECTRICITY

PART I.

CHAPTER I.

LAWS OF ELECTRICAL ACTION. POTENTIAL.

1. **ELECTRIFICATION.** If a carefully dried glass rod is held in the hand and rubbed with a piece of dry silk, it is found to acquire the property of being able to attract light bodies, such as little pieces of paper or straw, pith balls, &c. Several other bodies show the same property, as, for instance, a rod of ebonite rubbed with flannel or a cat's skin.

Bodies which possess this property are said to be *electrified*, or *electrically excited*, or to be charged with *Electricity*.

Among bodies which show electrical excitement well in this way, may be mentioned, besides glass and ebonite, amber, sulphur, resin, and sealing-wax.

2. With certain bodies this experiment is impossible, as, for instance, with a rod of brass or any other metal. Electricity is developed on the surface of the metal, as it is on the rod of glass, but it immediately flows away through the hand and body to the earth. But if the metal rod be provided with a handle

made of glass it will retain its electrical charge just as the glass rod did.

We must then distinguish between two classes of bodies :—

(i) Those which allow electrical charges readily to pass through them, or over their surfaces. These bodies are called *conductors*.

(ii) Those which offer good resistance to the passage of electrical charges, and retain them for a long time. These bodies are called *non-conductors*, or *insulators*.

No sharp line can be drawn between these two classes, as bodies possess all shades of conducting and insulating power. Also there exists no perfect conductor, and no perfect insulator. Every body conducts electricity, or allows it to pass, with more or less freedom; and the best-known conductor offers a certain amount of resistance to the passage.

3. Two kinds of Electrification. If we take two glass rods and rub them with a piece of silk, they will be found to exert a force of repulsion on each other. But a glass rod rubbed with silk and an ebonite rod rubbed with flannel are found to attract each other. A gilt pith ball suspended by a piece of silk and touched with the glass rod is found to be repelled by the glass, but attracted by the ebonite rod.

From these experiments we infer that there are two kinds of electricity. Now the glass rods were obviously charged alike, and on touching the pith ball with one of them it gave to it a part of its own charge. All these charges repel each other. The electrified ebonite rod behaves in a different way; it attracts any of these bodies. Thus we see that the ebonite rod is charged with a different kind of electricity. And we would further see that the ebonite rod would repel a similarly electrified ebonite rod, or a pith ball suspended by silk which had been touched by it.

We thus infer the existence of two kinds of electricity, and the mutual actions of bodies charged with them. This action may be stated in the following law of electrical action.

LAW I. **Bodies charged with like electrical charges repel each other; bodies charged with unlike electrical charges attract each other.**

These two sorts of electricity are called positive and negative electricities. That with which the glass rod was charged is called *positive* or *vitreous*. And that with which the ebonite was charged is called *negative* or *resinous*, as it is the kind of electricity a stick of resin would be charged with if rubbed with silk or flannel.

4. A sphere of metal, charged with electricity, can be made to give up half its charge by placing it in contact with an exactly equal metallic sphere, originally uncharged; both being far from all other charged or conducting bodies. From the symmetry of the arrangement the charge flows over the surfaces and divides itself equally between the two. The given sphere then has just half as much electricity on it as it had at first. In the same way we can divide the remaining half into two equal portions, and so on.

Since then we can say that one charge is $\frac{1}{2}$, $\frac{1}{4}$, and so on, of another, a charge of electricity is a quantity which we should be able to measure in terms of a fixed unit charge.

5. Coulomb has investigated with the torsion-balance the manner in which the force between two charged bodies depends on their charges, and on the distance between them. In the torsion-balance a gilt pith ball is carried on a light insulating horizontal arm, the whole being suspended at its centre of gravity by a vertical wire or glass thread. The ball is charged, and another similarly charged ball is brought up, repelling it, and put into the place it originally occupied. The balls must be allowed to be at a considerable distance from each other as compared with their dimensions. This distance may be adjusted by turning the top of the wire. The moment of the deflecting force is just balanced by the moment of torsion in the wire, and this is proportional to the amount of twist in the wire. The amount of twist may be measured, by observing the angles through which the top and bottom of the wire have turned, by means of scales provided for the purpose.

The fixed ball is then removed and made to give up half its charge to an exactly similar ball. To bring the moveable ball

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placement of the charge, whether along a conductor, or with the body on which it is. This may be shown as follows.

If the amounts of work were different in the two cases, by constantly allowing a charge to move from one position to a second with a conductor, and back again along a conducting wire, an indefinite supply of energy would be obtained.

Thus the energy expended when given charges move in any manner under the action of given charges, whether they move with charged bodies or along the bodies, can be calculated by the ordinary mechanical methods of calculating the work done by given forces.

7. When a conducting body is charged with electricity, which is at rest on it, it can be shown by experiment, and it will be proved later on, that the charge resides on its surface alone.

When a body is spoken of as being charged with electricity it is to be understood that the charge has taken up its state of equilibrium upon the body; with an ordinary conducting body, such as a metallic conductor, this will practically be the case in a very short time, a small fraction of a second, after it has received a charge.

When a body is charged with electricity, as a rule the surface-charge is not uniformly distributed all over its surface, but the quantity per unit area varies from point to point.

DEF.: **THE SURFACE DENSITY at any point on the surface of a charged body is the quantity of electricity per unit area at that point.**

Or, if we take a very small area s including the point, and e is the quantity of electricity on s, then the surface-density at the given point is the limiting value of e/s, when s is made indefinitely small.

DEF.: **THE VOLUME-DENSITY at a given point is the quantity of electricity per unit volume at that point.**

As has been said, the volume-density at all points within the mass of a conductor is zero.

8. **WORK.** Suppose a charge of electricity e concentrated at a point O. Let us consider the work done by it in ergs, or

centimetre-dynes, on an indefinitely small body with a positive unit charge, as this body moves from A to B, the distance OA being r_1 cms., and OB r_2 cms.

Let the body take any path s from A to B. Let ds be any elementary portion of this path; and r and $r + dr$ the distances of the body from O at the beginning and end of ds. The force on the body in the path ds is e/r^2. The resolved part of this in the direction of ds is $\frac{e}{r^2} \cdot \frac{dr}{ds}$.

The work done on the body in the path ds is

$$\frac{e}{r^2} \cdot \frac{dr}{ds} \cdot ds \,. = \frac{e}{r^2} \cdot dr.$$

Thus the entire work done is

$$\int_{r_1}^{r_2} \frac{e}{r^2} \cdot dr. = e\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

The work that would be done on the body by the charge e as it moves off from A to an infinite distance would be e/r_1.

The region we are considering as under the influence of any given electrical charges is called an **ELECTRICAL FIELD.**

DEF.: **THE POTENTIAL at any point of an electrical field is the work that would be done on an indefinitely small body charged with a positive unit of electricity, and supposed not to disturb the charges of the field, as it moves away from the given point to an infinite distance.**

If there are any number of charges q, q', q'', &c. in a field, supposed concentrated at points, and the point P is at distances r, r', r'', &c. from those points, the potential at P is

$$\frac{q}{r} + \frac{q'}{r'} + \frac{q''}{r''} + \text{\&c}.$$

Let there be any surface density σ in a field, and any volume density ρ. Denote the distance of a point P from any element of surface ds by r, and its distance from any element of volume dv by r'. Then the potential at P is given by the expression

$$\iint \frac{\sigma}{r} ds + \iiint \frac{\rho}{r'} dv.$$

If a point is such that on the whole the field does negative work on the body charged with a positive unit as it is moved off from the point to an infinite distance, that is, if positive work must be done on the body to move it off against the action of the field; then the potential of the point is negative. This is the case, for instance, when all the charges of the field are negative.

9. DEF.: **THE ELECTRICAL INTENSITY at any point is the force in dynes with which an indefinitely small body carrying a positive unit charge and placed at that point, being supposed not to disturb the charges of the field, would be acted upon.**

Let V be the potential at a point P. Take a point Q indefinitely near to P, so that the distance PQ is ds. Let the potential at Q be $V+dV$. Let F be the resolved part of the electrical intensity in the direction PQ. The work done by the field on an indefinitely small body with a positive unit charge, as it moves from P to Q, is the work that would be done on it as it moves from P to an infinite distance —the work that would be done on it as it moves from Q to an infinite distance, that is $V-(V+dV)$, or $-dV$. This work is also Fds. $\therefore F=-\dfrac{dV}{ds}$.

10. DEF.: **AN EQUIPOTENTIAL SURFACE is the locus of all points in an electrical field at which the potential has the same given value.**

DEF.: **A LINE OF FORCE is a line drawn in an electrical field such that its direction at any point is the direction of the electrical intensity at that point.**

Every equipotential surface is cut at right angles by all the lines of force which meet it.

For as a charged body is moved in any direction on an equipotential surface, no work is done on it, and therefore no force acts on it in the direction of its motion; that is, there is no component of force tangential to the surface. Therefore

the force is perpendicular to the surface; or the line of force cuts the surface normally.

When a set of electrical charges have taken up their state of equilibrium, the whole of any conductor, made throughout of the same material, is at the same potential. For if the potential did vary from point to point, there would be electrical forces urging positive electricity from places of high to places of low potential; and so a flow of electricity must take place in the conductor, or over its surface. But this cannot happen because the electricity is in a state of equilibrium. Therefore the whole conductor is in a state of equilibrium.

If, however, a conductor is made of two different materials, in general its two parts will be at different potentials; for the general effect of putting two bodies of different materials in contact is to cause a generation of electricity at the surfaces in contact, positive being formed on one body and negative on the other. The ordinary production of electricity by friction is a case of this, friction being used to bring large portions of the surfaces in contact with each other. In the case of two conductors friction is not necessary; they take their charges, and attain their full difference of potential, by being merely put into contact.

11. The surface of any conductor may be regarded as one of the equipotential surfaces of the field, and the lines of force passing away from the surface must pass away at right angles to the surface.

If a small positively charged body without mass were placed in an electrical field under the action of no forces but those of the field, it would move along a line of force in what we may call the positive direction of the line of force; a similar negatively charged body would move along a line of force in the negative direction.

An electrical field may be regarded as filled with lines of force, and equipotential surfaces cutting them at right angles.

As we pass along a line of force in its positive direction, we pass across an infinite number of equipotential surfaces, the

values of the potentials at which are constantly diminishing, algebraically.

No line of force can return on itself. For as we pass along a line of force in the positive direction we come to points of smaller and smaller potentials; so that we must be constantly coming to new regions.

No line of force can branch off into two lines at any point. For this would mean that at some point there are two directions of the resultant electrical intensity.

12. Proof of the Law of Force. The experiments of Coulomb with the torsion-balance can only be considered to prove the law of inverse squares in an approximate manner. The truth of the law was deduced by Cavendish from the fact that if a conductor is placed within a hollow closed conductor, and in electrical connexion with it, there is no charge on the inner conductor.

The following proof is taken from *Maxwell's Electricity and Magnetism.*

Suppose we have a conducting sphere enclosed in a hollow conducting sphere. If a charge is given to them when they are in electrical connexion, and then the connexion is broken, no trace of charge can be found on the inner sphere.

Let $\phi(r)$ be the law of force. That is, suppose that two indefinitely small charges, e and e', at a distance r apart, repel each other with a force $ee'\phi(r)$.

Then the potential due to a unit charge at a point distant r from it is

$$\int_r^\infty \phi(r)\,dr.$$

Let $f(r)$ be a function of r, such that $f'(r) = r\int_r^\infty \phi(r)\,dr.$

Now suppose we have a conducting sphere of radius a charged to surface density σ. Let us take a point P at a distance b from O, the centre of this sphere, and let r be the distance from P to a point Q on the sphere. Let the angle POQ be θ. An element of charge at Q may be denoted by $\sigma a^2 \sin\theta\, d\theta\, d\phi$.

And we have $\quad r^2 = a^2 + b^2 - 2ab\cos\theta.$

Potential at P due to element at Q is

$$\sigma a^2 \int_r^\infty \phi(r)\sin\theta\, d\theta\, d\phi\, dr = \sigma a^2 \frac{f'(r)}{r}\sin\theta\, d\theta\, d\phi.$$

But $\qquad r\,dr = ab\sin\theta\, d\theta.$

Thus potential due to charged sphere is

$$\int_{r_2}^{r_1}\int_0^{2\pi} \sigma \frac{a}{b} f'(r)\, dr\, d\phi,$$

$$= 2\pi\sigma \frac{a}{b}\left[f(r_1) - f(r_2)\right],$$

r_1 and r_2 being the greatest and least distances of P from the sphere.

Thus if α is the entire charge on the sphere we get for the potential at a point outside, inside, or on the sphere, the values

$$\frac{\alpha}{2ab}\left[f(a+b) - f(b-a)\right],$$

$$\frac{\alpha}{2ab}\left[f(a+b) - f(a-b)\right],$$

$$\frac{\alpha}{2a^2}\left[f(2a) - f(0)\right].$$

Now let our outer sphere have radius a, and inner one radius b. Let their charges be α and β, and their potentials A and B.

Then $A = \dfrac{\alpha}{2a^2}\left[f(2a) - f(0)\right] + \dfrac{\beta}{2ab}\left[f(a+b) - f(a-b)\right].$

$$B = \frac{\alpha}{2ab}\left[f(a+b) - f(a-b)\right] + \frac{\beta}{2b^2}\left[f(2b) - f(0)\right].$$

But when $A = B$, β is always 0.

$$\therefore\ b\left[f(2a) - f(0)\right] = a\left[f(a+b) - f(a-b)\right].$$

Differentiating twice with respect to b, and dividing by a, we get

$$f''(a+b) = f''(a-b).$$

$$\therefore\ f''(r) = C_0, \text{ some constant.}$$

$$f'(r) = C_0 r + C_1.$$

$$\therefore\ \int_r^\infty \phi(r)\, dr = C_0 + \frac{C_1}{r}.$$

$$\therefore\ \phi(r) = \frac{C_1}{r^2}.$$

CHAPTER II.

THEOREMS ON THE ELECTROSTATIC FIELD.

1. The following is a very important theorem:

In an electrical field let any closed surface S be described. Let N be the component of the electrical intensity at any element ds of the surface resolved normally to the surface, and outwards. Let M be the entire quantity of electricity inside the surface. Then the integral $\iint N\,ds$ taken all over the surface is equal to $4\pi M$.

Let us consider a quantity e of electricity, inside the surface S, and concentrated at a point O. Let r be the distance from O to the element of surface ds, θ the angle between r produced outwards, and the normal to the surface drawn outwards. Let N be the component of the electrical intensity, along this normal, due to e. Then the entire intensity is e/r^2.

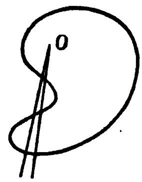
$$\therefore \quad N\,ds = \frac{e}{r^2}\cos\theta\,.\,ds.$$

And numerically $\cos\theta\,.\,ds$ is equal to the normal section at ds of the cone with vertex at O, and sides passing through the contour of ds. Thus $\cos\theta\,.\,ds/r^2$ is numerically equal to the solid angle subtended at O by ds. And this term will have the $+$ or $-$ sign according to the sign of $\cos\theta$, that is, according as O faces the inside or outside of ds.

Now a straight line drawn from O must meet the surface S an odd number of times. First it passes out of S. Then it may pass in, then out again, and so on, passing out the last time.

Suppose the cone with O as vertex and sides passing round

ds meets the surface S twice again. The first intersection of this cone with S contributes the positive term $e \cos\theta\, ds/r^2$ to the integral, and the other two intersections contribute two terms exactly equal in numerical value to this, but of opposite signs. In the same way, however many times this cone meets the surface after the first, all the terms contributed to the integral by these intersections after the first cut each other out two by two, and we get only the term $e \times$ solid angle of cone.



Thus when we take the entire integral all over the surface, we get for its value $e \times$ the sum of the solid angles of all the cones filling up the whole space about O, that is, $e \times 4\pi$.

If the charge e' were taken outside the surface S at a point O', any cone with vertex at O' and an indefinitely small solid angle must meet S an even number of times and contribute terms to the integral, which may all be seen as before to cut each other out two by two, each of these terms being of the form $e' \cos\theta'\, ds/r^2$, the letters having similar meanings to those already used.

Now the entire normal component N at any element of area ds, multiplied by ds, is made up of a number of terms such as $e \cos\theta\, ds/r^2$, and a number of terms such as $e' \cos\theta'\, ds/r'^2$. So that when we take the integral $\iint N ds$ all over the surface, we see that the first set of terms gives rise to $4\pi \times$ the sum of all the inside charges, and the second set gives rise to zero.

Thus $\iint N ds = 4\pi M.$

2. We can apply this proposition to prove several properties of electrical fields and conductors.

There is no charge within the body of a conductor. Let us consider any indefinitely small closed surface within the mass of the conductor. There is no electrical intensity at any point of this surface, since all points on it are within the conductor. Thus the integral $\iint N ds$ for this surface is zero. Therefore

the electrical charge within it is zero. Thus there is no electrical charge anywhere inside the mass of the conductor.

3. If any number of electrical charges are placed inside a cavity in a closed conductor, the algebraical sum of all these charges and of the charge of the inside surface of the cavity in which these charges are placed is zero.

Consider a closed surface traced out in the conductor so as to enclose the cavity. There is no electrical intensity anywhere on this surface. Thus for this surface $\iint N ds = 0$. Therefore the algebraical sum of all the charges inside it is zero. But these charges are only those placed inside the cavity and the distribution of electricity residing on the inside surface of the cavity.

4. POISSON'S AND LAPLACE'S EQUATIONS.

Let us take three rectangular axes, Ox, Oy, Oz, in the electrical field. Let V be the potential at a point P, whose coordinates are x, y, z. Through P draw three indefinitely short straight lines, dx, dy, dz, parallel to the axes, and let a rectangular parallelopiped be described on them as three adjacent sides. Let ρ be the electrical density at P. Then the quantity of electricity within $dx\,dy\,dz$ is $\rho\,dx\,dy\,dz$.

The electrical intensity perpendicular to the face $dy\,dz$ is $-\frac{dV}{dx}$. Thus the integral $\iint N ds$ for this face is $\frac{dV}{dx}dy\,dz$. That for the opposite face is $-\left(\frac{dV}{dx}+\frac{d^2V}{dx^2}dx\right)dy\,dz$. These two together give $-\frac{d^2V}{dx^2}dx\,dy\,dz$.

The other two pairs of faces give $-\frac{d^2V}{dy^2}dx\,dy\,dz$, and $-\frac{d^2V}{dz^2}dx\,dy\,dz$.

$$\therefore\quad -\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dx\,dy\,dz = 4\pi\rho\,dx\,dy\,dz,$$

or

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2} = -4\pi\rho,$$

This is called Poisson's equation.

If there is no electricity at the point P, we get

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0.$$

This is called Laplace's equation.

If P is within the body of a conductor,

$$\frac{d^2 V}{dx^2} = \frac{d^2 V}{dy^2} = \frac{d^2 V}{dz^2} = 0. \quad \therefore \ \rho = 0.$$

Thus there is no electricity within the body of a conductor (as already proved).

5. DEF.: **A TUBE OF FORCE is a tubular surface bounded by lines of force.**

Let us consider a tube of force of indefinitely small section. Let s and s' be its sections normal to its sides at two places. And let there be no electricity within the tube between s and s'. Let F be the electrical intensity at s, which we may consider to be constant all over s, and let F' be that at s'. There is no normal component of intensity anywhere on the sides of the tube, and F and F' are respectively perpendicular to s and s'. Thus, taking the surface integral of normal intensity all over the tube, we get $Fs = F's'$.

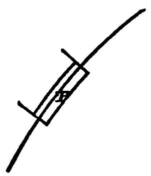
6. Let a number of lines of force be drawn across s, so that the number per unit area is proportional to the force at s. Let these be produced to s'. Then the intensity at any section, being inversely proportional to the section, is proportional to the number of lines of force across unit area at that section. So that if the number of lines of force represents the intensity at any section, it does so at every section of the tube. In this way we may represent the entire field, and the force at any point of it, by the frequency of the lines of force in it.

7. In the same way the intensity at any point of the field may be represented by the nearness of the equipotential surfaces, or the number passing across unit length of a line of force at the point. For if r is measured along a line of force, the intensity is $-\frac{dV}{dr}$. Let us then draw a number of equipotential surfaces corre-

sponding to values of the potential differing from each other by a constant quantity. The intensity at any point is inversely proportional to the distance between two equipotential surfaces at that point, or is proportional to the number of equipotential surfaces passing across unit length of a line of force at the point.

8. The electrical intensity just outside a charged conductor at a point at which the surface density is σ is $4\pi\sigma$.

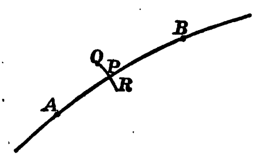
Let P be the point at which the surface density is σ.



Take a very small area s about P on the surface, and draw a tube of force whose sides all pass through the contour of s. Close up this tube by a surface drawn just outside the surface of the given conductor and parallel to s, and therefore equal to s. For we may regard this tube as a cylinder, and the two surfaces, s, and that drawn parallel to it, as two planes perpendicular to its sides. Produce the tube into the conductor, and close it up by another surface parallel to s. Considering the entire closed surface thus formed, the only portion of it over which there is any normal component of intensity is the surface drawn parallel to s outside the conductor. Let F be the electrical intensity just outside s. Then for the entire closed surface $\iint N ds$ is Fs. And the charge within the surface is σs.

$$\therefore \quad Fs = 4\pi\sigma s. \qquad \therefore \quad F = 4\pi\sigma.$$

9. At a point on the surface of a conductor where the electrical surface density is σ, the surface is under a stress of $2\pi\sigma^2$ dynes per square centimetre.



Let P be the point under consideration. About P describe a small circular area AB, so small that we may consider it a plane. On the normal at P, just outside, and just inside the conductor, take two points, Q and R, equally distant from P, and such that QR is inde-

finitely small compared with the dimensions of AB. So that Q and R are practically situated in just the same way with regard to all other charges in the field except the distribution on AB.

Now let us consider the parts of the electrical intensity at Q, due to the distribution on AB, and due to the rest of the field. The entire intensity at Q is along the normal at P. And that due to the distribution on AB is evidently along the same normal. So that that due to the rest of the field is in the same direction. Let F be the part of the intensity due to AB, and F' that due to the rest of the field. Then the intensity at Q is $F'+F$, and that at R is $F'-F$. Now the intensity at Q is $4\pi\sigma$, and that at R is 0.

$$\therefore\ F'+F = 4\pi\sigma, \text{ and } F'-F = 0.$$
$$\therefore\ F' = 2\pi\sigma.$$

Thus an indefinitely small portion of surface s, which we may consider plane, carrying a charge $s\sigma$, is urged by all the other charges in the field along the normal to the surface with a force of $2\pi\sigma^2 s$ dynes. Or the stress is along the normal outwards, and equal to $2\pi\sigma^2$ ~~degrees~~ dynes per square centimetre.

10. With regard to the statement that the action of all the charges in the field, excluding the distribution on AB, is the same at P and at Q, the following remarks may be made.

Suppose a charge e placed at O, the origin of coordinates. Then x-component of the electrical intensity at (x, y, z) is

$$\frac{e}{x^2+y^2+z^2} \cdot \frac{x}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{e\,.\,x}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

The intensity at $(x+h,\ y+k,\ z+l)$ has for its x component

$$\frac{e\,(x+h)}{\{(x+h)^2+(y+k)^2+(z+l)^2\}^{\frac{3}{2}}};$$

and if h, k, l are indefinitely small as compared with x, y, z, this only differs from $\dfrac{e\,x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ by a quantity indefinitely small as compared with this last quantity. In the same way if we take

any finite distribution of electricity which we may consider made up of infinitesimal charges collected at points all of which are at distances from (x, y, z), compared with which h, k, l are indefinitely small, the differences of the resolved parts in any direction of the electrical intensity at (x, y, z) and at $(x+h, y+k, z+l)$ give rise to an indefinitely small quantity of a higher order than the distance of (x, y, z) from any part of the electrical distribution in the field.

11. The stress per unit area on the charged surface, which has for expression $2\pi\sigma^2$, may also be expressed in terms of F, the electrical intensity just outside the surface. For $F = 4\pi\sigma$. Thus the stress per unit area is $F^2/8\pi$.

Let $\frac{dV}{dn}$ denote the variation of the potential per unit distance at the surface of a conductor, along the normal to the surface measured outwards. Then, since

$$F = -\frac{dV}{dn} \text{ and } F = 4\pi\sigma; \quad \therefore \sigma = -\frac{1}{4\pi}\frac{dV}{dn}.$$

And the stress per unit area is $\frac{1}{8\pi}\left(\frac{dV}{dn}\right)^2$.

12. Consider a very narrow tube of force. Let s be its cross-section at any place, and F the electrical intensity at s. We have seen that as we pass along the tube, as long as we come to no electrical charges within it, the product Fs remains constant. Thus as we pass along the tube in either direction, as long as we come to no electrical charges, we come to new regions in which electrical force is being exerted, and in which therefore the tube is continued. Thus a tube of force, and therefore a line of force, cannot begin or end except on a surface charged with electricity.

Since the electrical intensity, expressed in terms of the electrical density, just outside a surface is given by the equation $F = 4\pi\sigma$, a line of force must begin (being traced in its positive direction) on a positively charged surface, and end on a negatively charged surface.

13. The quantities of electricity on the two portions of

charged surface of conductors forming the ends of a tube of force are equal and of opposite signs.

For if we consider the tubes produced slightly into the conductors and closed up in them we get formed thus a closed surface, over the whole of which there is no normal component of electrical intensity. Thus the algebraical sum of the charges inside this surface is zero.

CHAPTER III.

SYSTEMS OF CONDUCTORS.

1. Electrical Induction. If we have a conductor with a charge on it in electrical equilibrium, and other charges are brought into the field, the distribution of charge on the given conductor will, as a rule, be altered. For the introduction of the new charges into the field will alter the potentials of the various points of the given conductor in different degrees as a rule, and for equilibrium of the charge on the conductor all points on it must be at the same potential. So that the charge on it will distribute itself so as to produce the same potential all over the given conductor.

This disturbance of the charge on the conductor by the presence of new charges is called **Electrical Induction.**

2. The general effect of induction may easily be seen to be this. The approach of a positively charged body to a conductor will algebraically diminish the electrical density on the parts of it near the charged body, and increase the density on the parts of it far from the charged body; while a negatively charged body will act in just the opposite way.

3. The earth, and any conductor in electrical connexion with it, may be considered to be at zero potential. For the earth may be taken as a conductor in electrical equilibrium, and the potential at its centre, given by the formula $\Sigma \frac{q}{r}$, vanishes on account of the great distances of the centre from all the charges producing the potential. Thus all the earth and any conductor connected with it are at zero potential.

4. There is only one possible state of distribution of a given set of charges on the conductors in an electrical field.

First, we shall show that if the electrical density at every point in an electrical field is altered in any ratio, we get a distribution of electricity in equilibrium. For the potential at each point is changed in the same ratio. And since the potential all over each conductor was constant, the potential all over each conductor is still constant. Therefore the new distribution is in electrical equilibrium.

For the same reason, if there are two distributions, each of which is in equilibrium when existing on a given system of conductors, by superposing them we get a distribution in equilibrium.

Next, we shall show that if in an electrical field there is any number of conductors, the entire quantity of electricity on each being zero, and if there are no other charges in the field, then each conductor is in the neutral state, that is, the density at each point of it is zero.

Denote the conductors by $A_1, A_2, A_3, \ldots$. And let their potentials be $V_1, V_2, V_3, \ldots$, the potentials being arranged in algebraically descending order of magnitude.

Now no lines of force can end on A_1, for that would mean that there was a charge in the field at a potential higher than V_1 for a line of force ending on A_1 to start from. So that any lines of force meeting A_1 must start from it. Thus there can be no negative electricity on A_1 and the entire charge on A_1 is zero, therefore the density at every point of A_1 is zero.

There being no electricity on A_1, we may consider A_1 to be removed from the field, and by reasoning in the same way we find the density at every point of A_2 to be zero. And in the same way for each of the others.

Now suppose, if possible, that in a given electrical field, with given charges on the conductors, there could be two different distributions of the electricity on the conductors for electrical equilibrium. Let the density at any point of a conductor be denoted by σ and σ' in the two cases.

Let us take the second distribution, and reverse its sign, and superpose it on the first. Then we get a distribution in the field in equilibrium. But there is now no charge in the field except on conductors. And each conductor has, on the whole, zero charge. Thus the density at every point of each conductor must be zero. Therefore σ' must be the same as σ. Or there is only one possible distribution for the given charges on the conductors.

5. Suppose we have a system of conductors $A_1, A_2, A_3, \ldots A_n$ with charges $Q_1, Q_2, Q_3, \ldots Q_n$, and at potentials $V_1, V_2, V_3, \ldots V_n$.

We shall show that the potential of any conductor can be expressed as a linear function of the n charges as follows:

$$V_r = p_{r1}Q_1 + p_{r2}Q_2 + \cdots p_{rr}Q_r + \cdots + p_{rn}Q_n.$$

The coefficients p_{r1}, &c. depend only on the form, size, and relative position of the conductors. Each p has two suffixes, the first relating to the conductor in the expression for whose potential it occurs, and the second to the conductor whose charge it multiplies.

First, suppose all the conductors to be discharged. Now give A_1 unit charge, and suppose this produces in the various conductors the potentials

$$p_{11}, p_{21}, p_{31}, \cdots p_{n1}.$$

If we give A_1 a charge Q_1, the potentials will be

$$p_{11}Q_1, \ p_{21}Q_1, \ p_{31}Q_1, \cdots p_{n1}Q_1.$$

Similarly we may suppose that by giving to A_2 the charge Q_2, the others being uncharged, the conductors are raised to potentials $\quad p_{12}Q_2, p_{22}Q_2, p_{32}Q_2 \cdots p_{n2}Q_2,$

p_{12}, p_{22}, p_{32}, &c. being constants for the given conductors.

In the same way charge each of the conductors separately, all the others being at the same time uncharged, and we get similar expressions for the potentials.

Now if we charge all the conductors simultaneously with the charges $Q_1, Q_2, Q_3, \ldots$, we get an electrical distribution which is merely the superposition of all the distributions obtained on all

the conductors, by charging them separately with the charges Q_1, Q_2, Q_3, ... And the potential at any point is the sum of the several potentials obtained from these separate charges.

Thus we will have for the potentials V_1, V_2, V_3, ...,

$$V_1 = p_{11}\,Q_1 + p_{12}\,Q_2 + \dots + p_{1r}\,Q_r + \dots + p_{1n}\,Q_n.$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$V_r = p_{r1}\,Q_1 + p_{r2}\,Q_2 + \dots + p_{rr}\,Q_r + \dots + p_{rn}\,Q_n.$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$V_n = p_{n1}\,Q_1 + p_{n2}\,Q_2 + \dots + p_{nr}\,Q_r + \dots + p_{nn}\,Q_n.$$

6. By solving these n linear equations we can plainly express the charges as linear functions of the potentials. Thus

$$Q_1 = q_{11}\,V_1 + q_{12}\,V_2 + \dots + q_{1r}\,V_r + \dots + q_{1n}\,V_n.$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$Q_r = q_{r1}\,V_1 + q_{r2}\,V_2 + \dots + q_{rr}\,V_r + \dots + q_{rn}\,V_n.$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$Q_n = q_{n1}\,V_1 + q_{n2}\,V_2 + \dots + q_{nr}\,V_r + \dots + q_{nn}\,V_n.$$

The p's are called coefficients of potential, and the q's coefficients of induction.

Any coefficient of induction having its two suffixes the same, such as q_{rr}, is called the capacity of the corresponding conductor, A_r, with respect to the system.

7. Any two coefficients of potential with the same suffixes, only reversed, are equal. That is, $p_{rs} = p_{sr}$.

This may be proved by means of the following theorem of Gauss.

If any distribution of masses m_1, m_2, m_3, ... produces a potential denoted at any point by V, and any distribution of masses m_1', m_2', m_3', ... produces a potential V', then $\Sigma m V' = \Sigma m' V$.

That is, the sum of each elementary mass of one system multiplied by the potential produced, at the point where the elementary mass is, by the other distribution is equal to the same sum taken for the second system of masses and the potential due to the first distribution.

The theorem is merely an identity, and its truth is seen by

writing for V' its value, which is the sum of each elementary mass m' divided by its distance from m, and a similar expression for V. Then, denoting any distance between one of the masses m, and one of the masses m' by r, each side of the equation becomes $\Sigma \frac{mm'}{r}$. Thus the two expressions are identical.

8. To apply this theorem, let us consider as our two distributions m and m', those which arise from a unit charge given to A_r, and from a unit charge given to A_s, respectively.

The part of $\Sigma m V'$ arising from any conductor except A_r is zero, because the potential is the same all over any such conductor, and its entire charge is zero. Thus $\Sigma m V'$ becomes p_{sr}, and $\Sigma m' V = p_{rs}$; $\therefore p_{rs} = p_{sr}$.

9. The two coefficients of induction, q_{rs}, q_{sr}, are equal.

Consider the two electrical distributions in which A_r is raised to unit potential, and in which A_s is raised to unit potential, all the others in both cases being at potential zero. And let us apply the theorem of Gauss to these distributions. Then $\Sigma m V' = q_{rs}$; and $\Sigma m' V = q_{sr}$: $\therefore q_{rs} = q_{sr}$.

10. Energy of charging a system of conductors.

Let us suppose the system to pass from the state in which the charge and potential of any conductor are denoted by the letters Q, V to that in which they are denoted by the letters Q', V'. And suppose each charge Q increases at a rate proportional to its final total increment $Q'-Q$. So that the charge of a conductor may be expressed at any instant as $Q + x\,(Q'-Q)$, x being the same for each conductor at each instant, and increasing from 0 to 1, and its potential therefore as $V + x\,(V'-V)$.

(1) Let us suppose that the charging is done by bringing each element of charge dQ from an infinite distance up to the corresponding conductor at potential V. The work done in this operation is VdQ. Thus the energy expended on any conductor is

$$\int_0^1 \{V + x\,(V'-V)\}(Q'-Q)\,dx = \tfrac{1}{2}\,(Q'-Q)\,(V'+V).$$

And the entire energy expended is

$$\tfrac{1}{2}\Sigma(Q'-Q)(V'+V).$$

(2) Let us suppose that the charging is done by carrying the entire system up to each element of charge originally at an infinite distance off. In this operation let a conductor having a charge Q have its potential increased by dV. Thus the work done on this conductor in this operation is $Q\,dV$. And the entire work done on any conductor

$$\int_0^1 \{Q+x(Q'-Q)\}(V'-V)\,dx = \tfrac{1}{2}(V'-V)(Q'+Q).$$

And the entire energy expended is

$$\tfrac{1}{2}\Sigma(V'-V)(Q'+Q).$$

Thus we have for the energy expended two expressions;

(1) **The sum of the products of each increment of charge into the mean of the corresponding potentials.**

(2) **The sum of the products of each increment of potential into the mean of the corresponding charges.**

11. **Energy of charging the system of conductors.** Two expressions can be found for the energy of charging in terms of the coefficients p and q.

In both cases let us suppose all the conductors charged at a uniform rate, which is proportional for each conductor to the final charge the conductor has on it. So that at any time a conductor which is finally to have charge Q and be at potential V may be supposed to have charge $x\,Q$ and be at potential $x\,V$.

The charges being $x\,Q_1$, $x\,Q_2$, $x\,Q_3$, ... the potential of A_1 is

$$p_{11}x\,Q_1+p_{12}x\,Q_2+p_{13}x\,Q_3+\ldots.$$

Therefore the energy expended in charging A_1 is

$$\int_0^1 (p_{11}\,x\,Q_1+p_{12}\,xQ_2+p_{13}\,x\,Q_3+\ldots)\,Q_1\,dx$$
$$= \tfrac{1}{2}(p_{11}\,Q_1^2+p_{12}\,Q_1\,Q_2+p_{13}\,Q_1\,Q_3+\ldots).$$

And we have similar values for each of the others.

Remembering that $p_{rs}=p_{sr}$, we get for the entire energy expended on the system,

$$\tfrac{1}{2}p_{11}Q_1^2+\tfrac{1}{2}p_{22}Q_2^2+\dots+p_{12}Q_1Q_2+p_{13}Q_1Q_3$$
$$+p_{23}Q_2Q_3+\dots.$$

Again, the potentials at any instant being $xV_1, xV_2, xV_3, \dots$ the charge on A_1 is

$$q_{11}xV_1+q_{12}xV_2+q_{13}xV_3+\dots.$$

The element of charge is

$$(q_{11}V_1+q_{12}V_2+q_{13}V_3+\dots)\,dx.$$

Thus the energy expended in charging A_1 is

$$\int_0^1 xV_1(q_{11}V_1+q_{12}V_2+q_{13}V_3+\dots)\,dx$$
$$=\tfrac{1}{2}(q_{11}V_1^2+q_{12}V_1V_2+q_{13}V_1V_3+\dots).$$

And similar values for each of the others.

Remembering that $q_{rs}=q_{sr}$, we get for the entire energy expended on the system,

$$\tfrac{1}{2}q_{11}V_1^2+\tfrac{1}{2}q_{22}V_2^2+\dots$$
$$+q_{12}V_1V_2+q_{13}V_1V_3+q_{23}V_2V_3+\dots.$$

12. Alternative proofs that $p_{rs}=p_{sr}$ and $q_{rs}=q_{sr}$.

We have the two expressions for the infinitesimal increment of energy δW of a system, δQ and δV denoting infinitesimal increments of charge and potential,

$$\delta W=\Sigma(V\delta Q)=\Sigma(Q\delta V).$$

Thus if W_Q and W_V denote the energy of the system expressed in terms of the charges and potentials respectively, we have

$$V_r=\frac{dW_Q}{dQ_r},$$
$$Q_r=\frac{dW_V}{dV_r}.$$

Then from the expressions for the potentials in terms of the charges and the coefficients p, we have

$$p_{rs}=\frac{dV_r}{dQ_s}=\frac{d}{dQ_s}\cdot\frac{dW_Q}{dQ_r}=\frac{d}{dQ_r}\cdot\frac{dW_Q}{dQ_s}=\frac{dV_s}{dQ_r}=p_{sr}.$$

And from the expressions for the charges in terms of the potentials and the coefficients q, we have

$$q_{rs} = \frac{dQ_r}{dV_s} = \frac{d}{dV_s} \cdot \frac{dW}{dV_r} = \frac{d}{dV_r} \cdot \frac{dW_V}{dV_s} = \frac{dQ_s}{dV_r} = q_{sr}.$$

These proofs are from Maxwell's *Electricity and Magnetism.*

13. Let A_r have a unit charge, all the other conductors being uncharged. Then the potential of A_r is p_{rr}. The potentials of the other conductors are $p_{1r}, p_{2r}, \ldots, p_{rs}, \ldots$. Let A_s be that one of the other conductors which has the greatest potential. Now A_s has positive and negative electricity on its surface, the two being numerically equal in quantity. Therefore some lines of force end on A_s. And none of these come from any of the other conductors, excluding A_r, because these are all at lower potentials than A_s. Now no lines of force can pass from infinity into the field. For if a very large sphere of radius r is described so as to have all the conductors practically at its centre, the electrical intensity at every point of its surface tends outwards and is $\frac{1}{r^2}$. So that at all points on this sphere the lines of force are passing from the field to infinity. Thus the lines of force that end on A_s must start from A_r. A_r then has a higher potential than A_s. Therefore p_{rr} is $> p_{sr}$. **Therefore p_{rr} is greater than any of the other coefficients p_{1r}, $p_{2r}, \ldots$.**

14. The lines of force passing off from the conductor at lowest potential must pass away to infinity, where the potential is zero. Therefore this conductor must be at a positive potential.

Therefore all the coefficients $p_{1r}, p_{2r}, p_{3r}, \ldots$ are positive.

15. Let A_r be raised to unit potential, and all the other conductors be at zero potential. No lines of force can pass away from any other conductor, as A_s, for none can go from A_s to infinity where the potential is zero, and none can go to A_r where the potential is unity. Thus lines of force pass from A_r to the other conductors, and as a rule some pass from A_r to infinity. Therefore all the electricity on A_r is positive, and all that on the other conductors is negative.

Therefore q_{rr} is positive, and all the coefficients such as q_{sr} are negative.

16. If some lines of force pass from A_r to infinity, which will be the case if none of the other conductors surrounds A_r, the negative electricity on the others is less than the positive on A_r. **Then** $q_{rr} > -(q_{1r}+q_{2r}+\ldots+q_{sr}+\ldots)$.

17. Suppose A_r with some of the other conductors enclosed in A_s, which is a hollow closed conductor. Then all the lines of force starting from A_r must stop on A_s and the other conductors inside A_s. So that the electricity on A_r is just equal in magnitude but of opposite sign to all that on A_s and the other conductors inside A_s. Also there is no charge on any conductor outside A_s, for no lines of force can start from or stop on any such conductor. **Thus q_{rr} is equal to the sum of all the coefficients with suffix r relating to A_s and the other conductors inside it, taken with the — sign. And the coefficients with suffix r relating to the conductors outside A_s are all zero.**

18. If A_r is outside A_s, a hollow closed conductor, and some other conductors are inside A_s, we can show in the same way that **the coefficients with suffix r relating to all the conductors inside A_s are zero.**

19. It should be noticed that a coefficient such as p_{rs} or q_{rs} does not depend only on the conductors A_r and A_s, but also depends, as a rule, on all the others. Also the coefficients p_{rr}, q_{rr} depend on the entire system of conductors.

20. In any charged electrical system we have seen that the electrical energy of the system is $\frac{1}{2}\Sigma(QV)$.

Now let the system undergo any change of configuration, all the charges remaining the same as they were at first. The potentials will as a rule vary, and the increment of energy of the system will be $\frac{1}{2}\Sigma(Q\delta V)$.

If this change of configuration is produced by the electrical forces of the system, the quantity $\frac{1}{2}\Sigma(Q\delta V)$ will be negative, and numerically equal to the positive work done by these forces.

21. Now let us suppose that this change in the configuration

is indefinitely small, and let us consider what the variation in the energy of the system will be when the same change in the configuration is produced, the *potentials* of the various conductors of the system being kept constant. In this case, as the conductors move, their potentials tend to vary; and to keep them constant the charges have to be varied; thus electrical energy, positive or negative, must be supplied to the system.

Let the potentials be given in terms of the charges by the ordinary equations, such as

$$V_r = p_{r1} Q_1 + \dots + p_{rs} Q_s + \dots.$$

Now the infinitesimal variations in the configuration will produce infinitesimal variations dp_{r1}, &c. in the coefficients p.

We have for the energy of the system the expression

$$W = \dots + \tfrac{1}{2} p_{rr} Q_r^2 + \dots + p_{rs} Q_r Q_s + \dots.$$

Let us denote the infinitesimal increments of the energy of the system produced in the given variations of the configuration, when the charges remain constant, and when the potentials remain constant, by $d\,W_Q$ and $d\,W_V$ respectively.

Then we have

$$d\,W_Q = \tfrac{1}{2} \Sigma\,(dp_{rs} Q_r Q_s).$$

In this expression the two suffixes r and s may become equal, and the coefficient p_{sr} is to be taken as well as p_{rs}.

Now if the potentials remain constant, we get by differentiating the expression for V_r, and multiplying it by Q_r,

$$0 = \dots + dp_{rs} Q_r Q_s + p_{rs} Q_r dQ_s + \dots.$$

In this the suffix s has a value corresponding to each conductor, the value r being included.

Adding all such equations as this, got by giving all values to r, we get

$$\Sigma\,(dp_{rs} Q_r Q_s) + \Sigma\,(p_{rs} Q_r\, dQ_s) = 0.$$

And since $p_{rs} = p_{sr}$, the group of terms from the second summation forming the coefficient of dQ_s is equal to V_s.

Therefore

$$\Sigma\,(p_{rs} Q_r\, dQ_s) = \Sigma\,(V_s dQ_s) = 2\,d\,W_V.$$

And we have seen that

$$\Sigma (dp_{rs} Q_r Q_s) = 2\,d W_Q.$$
$$\therefore \quad d W_Q + d W_V = 0.$$

If then the system is allowed to undergo any infinitesimal change of configuration under the action of the electrical forces, the potentials being kept constant, the energy of the system is increased by the same amount as it would have been diminished by if the charges had remained constant, and the potentials had been allowed to vary. The sources which keep the potentials constant are drawn upon for a quantity of energy double of this, one-half going to increase the energy of the system, and the other corresponding to the work done in the system.

It follows that if the system undergoes a finite change of configuration with constant potentials, and if W be the work done in the system by the electrical forces, the increase of energy of the system will be W, and the energy supplied by the sources will be $2\,W$. But in this case the work done, with charges constant, for the same configuration, and consequently the loss of energy, with charges constant, is not represented by W.

22. Let ϕ be any variable on which the configuration of the system depends. If ϕ is a line, the force tending to vary it is $-dW_Q / d\phi$. If ϕ is an angle, the moment of the forces tending to vary it is $-dW_Q / d\phi$. And we see that each of these expressions is also equal to $dW_V/d\phi$. Thus the *force* (using the word in its generalized sense) tending to increase the coordinate ϕ, of any type, is $d\,W_V/d\phi$.

23. If there is a hollow closed conductor with no charged bodies inside it, then the potential is the same throughout the inside region as that of the conductor, and there is no electricity on the inside surface.

There can be no line of force inside the conductor, for a line of force must start on a surface of one potential, and stop on a surface of a lower potential. Thus there is no variation of potential, or the potential is the same throughout the region inside as it is on the conductor itself. And every charged

surface is met by lines of force. Therefore there is no electricity on any part of the inside surface.

24. If a conductor is at zero potential, and there are no other charged bodies in the field, the potential is zero throughout the field, and there is no electricity on the outside surface of the conductor.

There can be no line of force in the field, starting and stopping in it, for in that case there must be two charges in it at different potentials. And there can be no line of force having one end on the surface of the given conductor and the other at infinity, where the potential is zero, because the potential of the given conductor is zero. Therefore there is no variation of potential in the field, or the potential is everywhere zero. And there is no electricity on the surface of the given conductor.

25. Suppose we have a hollow closed conductor kept at a constant potential V, with any number of charged conductors inside. Then, however the outside electrical distributions may be altered, the potentials and distributions inside are unaffected.

We shall include the distributions on the outside surface of the closed conductor among the outside distributions, and that on the inside surface among the inside distributions.

Now the given outside and inside distributions are in equilibrium, and keep the closed conductor at potential V.

Let us suppose these removed, and another set of outside distributions introduced, the conductor being kept at potential 0. Then there is no inside distribution at all. And we have the conductor and every point inside it at potential 0, under the action of the new outside distribution.

Now let us superpose these two states in equilibrium. We thus have the old and new outside distributions, and the old inside distributions in equilibrium, and the conductor at potential V, and the potential at every point inside just what it was at first.

Thus whatever new outside distributions are introduced, that is, however the outside distributions are changed, the only state

of equilibrium of the inside distributions is the original state. That is, the original state is unaltered. And the potential at every point inside remains what it was before.

26. By similar reasoning it can be seen that if the closed conductor is not kept at a constant potential, an alteration of the outside distributions will not alter the inside distributions, but will alter the potential at every point inside by the same quantity.

27. If a hollow closed conductor is kept at a constant potential V, no alteration of the inside distributions will alter the distribution or potential at any point outside.

For the given inside and outside distributions are in equilibrium, and keep the conductor at potential V.

Suppose these removed, and another set of inside distributions introduced, the conductor being kept at potential o. Then every point outside the conductor is at potential o, and there is no outside distribution at all.

Now let us superpose these two states in equilibrium. We see then that keeping the conductor at potential V, and with any set of inside distributions, the only state of equilibrium of the outside distributions is the original state.

CHAPTER IV.

CAPACITIES.

1. DEF.: **THE CAPACITY OF A CONDUCTOR is the charge which must be given to it to raise it to unit potential.**

When the capacity of a given conductor is spoken of without reference to any other conductors, it is supposed that there are no other charges or conductors in the field. Let then C be the capacity of a conductor, and suppose a charge Q is given to it raising it to potential V. We have the relation

$$Q = C V.$$

2. A condenser is an arrangement of two conductors, each of which presents a large portion of surface close to the surface of the other conductor. In this way, by keeping one at zero potential, a large charge of electricity can be given to the other without greatly raising its potential. For the potential of the other is determined by the charge on it of one sign, and the charge of the other sign that it induces on the conductor at zero potential. And these two to a great extent cancel each other's effect.

Let A_1 and A_2 be the two conductors forming the condenser, q_{11}, q_{12}, q_{22} their coefficients of induction. Then we have the ordinary equations

$$Q_1 = q_{11} V_1 + q_{12} V_2,$$
$$Q_2 = q_{12} V_1 + q_{22} V_2.$$

By arranging the two conductors as described we can make the quantities q_{11}, $-q_{12}$, q_{22} very large and nearly equal. We know that if A_1 is completely surrounded by A_2, $q_{11} = -q_{12}$.

And, without making A_2 enclose A_1, by putting the surfaces of A_1 and A_2 close to each other for a large portion of their extent when A_1 is charged to unit potential, and A_2 is at zero potential, most of their lines of force meet them both, and the quantities of electricity on them are nearly equal, and of opposite sign. Thus we have q_{11} nearly $= -q_{12}$, and similarly q_{22} nearly $= -q_{12}$.

DEF.: **THE CAPACITY OF A CONDENSER is the charge that must be given to one of its conductors to raise it to unit potential, the other being kept at zero potential.**

If A_1 and A_2 are the two conductors, we see from the above equations that the capacity of the condenser will be q_{11} or q_{22} according as A_2 or A_1 is the conductor kept at zero potential.

A simple form of condenser consists of two parallel conducting plates. If these are exactly equal and similar and similarly situated with regard to each other, it is obvious that $q_{11} = q_{22}$, and the capacity will be the same whichever conductor we suppose to be the one kept at zero potential. But it should be noticed that this depends too on the supposition that neither conductor is near enough to any other conductor or charged body to be influenced by it.

A common way of making a condenser is to have each of the conductors A_1 and A_2 composed of a large number of parallel plates all electrically connected. The plates of A_1 lie between those of A_2, and they all lie quite close together, adjacent ones being insulated from each other by some highly insulating substance, such as paraffined paper.

By making a condenser in this way we get a very high capacity. Also, since each one practically encloses the other, we have very nearly $q_{11} = -q_{12} = q_{22}$. And it is immaterial which conductor we use as the one kept at zero potential.

Let C be the capacity of the condenser. Then we have from the above equations

$$Q_1 = \quad C(V_1 - V_2),$$
$$Q_2 = -C(V_1 - V_2).$$

Thus when a condenser is made in this way we can say that its capacity is the numerical value of the charge which each

conductor has on it when there is unit difference of potential between them.

3. We shall now determine the capacities of some particular cases of condensers.

(1) Two concentric spheres of radii a and b.

Suppose the inner sphere of radius a to have a charge Q_1 and the other to be at zero potential. This will have a charge Q induced on its inner surface. The potential at the centre of the inner sphere is

$$V = Q\left(\frac{1}{a} - \frac{1}{b}\right);$$

$$\therefore \text{ the capacity is } C = \frac{1}{\frac{1}{a} - \frac{1}{b}} = \frac{ab}{b-a}.$$

(2) Two portions each of area S, opposite to each other, of two indefinitely extended parallel plates, at a distance a apart.

Let the two plates be charged with equal quantities of electricity of opposite signs. These will obviously distribute themselves over the two plates in the same manner, and the plates will have equal and opposite potentials. Let these be V and $-V$.

Now at points on the plates far away from the edges the action of the edges may be neglected, and the lines of force may be considered to be straight and perpendicular to the plates in the space between them, and the electrical density to be uniform.

Also at a portion of the surface on the back of one of the plates far from the edges, the lines of force start off at right angles to the surface and go for a long way approximately in straight lines at right angles to the surface. The electrical intensity just outside this surface is therefore very small, and therefore also the electrical density. Thus taking a portion of surface of the back of one of the plates far away from the edges, we may suppose the surface-density on it to be zero.

We may then consider two plane distributions σ and $-\sigma$ on two indefinitely extended planes. The potential V at any point of the positive plate is given by

$$V=\int_0^\infty\left(\frac{2\pi r\sigma dr}{r}-\frac{2\pi r\sigma dr}{\sqrt{r^2+a^2}}\right)$$

$$=2\pi\sigma\int_0^\infty\left(1-\frac{r}{\sqrt{r^2+a^2}}\right)dr$$

$$=2\pi\sigma\left[r-\sqrt{r^2+a^2}\right]_0^\infty$$

$$=2\pi a\sigma,$$

the expression vanishing at the upper limit.

In the same way

$$-V=-2\pi a\sigma.$$

Now the quantity of electricity on the portion S of the positive plate is

$$Q=S\sigma.$$

Thus the capacity is $\dfrac{Q}{2V}=\dfrac{S}{4\pi a}$.

(3) The portions cut off from two indefinitely long coaxal cylinders of radii a and b by two planes perpendicular to the axis and at distance l apart.

The lines of force are perpendicular to the axis, and the quantities of electricity opposite to each other on two corresponding elementary rings cut off by the same two planes perpendicular to the axis are equal and of opposite sign. Let the outer cylinder be at potential zero. Then there is no electricity on the outside of the outer cylinder at a distance from the edges. The potential V of the inner cylinder is the same as that at any point on the axis. Let σ be the density on the inner cylinder. Then

$$V=2\int_0^\infty\left(\frac{2\pi a\sigma dz}{\sqrt{a^2+z^2}}-\frac{2\pi a\sigma dz}{\sqrt{b^2+z^2}}\right)$$

$$=4\pi a\sigma\left[\log\frac{z+\sqrt{a^2+z^2}}{z+\sqrt{b^2+z^2}}\right]_0^\infty$$

$$=4\pi a\sigma\log\frac{b}{a}.$$

And the quantity of electricity on the portion of the inner cylinder is

$$Q = 2\pi a l \sigma.$$

$$\therefore \text{ the capacity is } \frac{l}{2\log\frac{b}{a}}.$$

4. These capacities may also be determined by the application of Laplace's equation

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0.$$

(1) For the sphere. If we transform Laplace's equation to ordinary polar coordinates, and remember that $\frac{dV}{d\phi} = 0$ and $\frac{dV}{d\theta} = 0$, we get the equation

$$r\frac{d^2 V}{dr^2} + 2\frac{dV}{dr} = 0.$$

This gives on integrating,

$$V = A + \frac{B}{r}.$$

Let V_1 and V_2 be the potentials of the spheres of radii a and b respectively, σ the density on the first sphere.

At the surface of the first sphere $\frac{dV}{dr} = -4\pi\sigma.$

$$\therefore \quad \frac{B}{a^2} = 4\pi\sigma,$$

$$\therefore \quad V_1 = A + \frac{4\pi\sigma a^2}{a},$$

$$V_2 = A + \frac{4\pi\sigma a^2}{b}.$$

$$\therefore \quad V_1 - V_2 = 4\pi\sigma a^2\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= Q\left(\frac{1}{a} - \frac{1}{b}\right),$$

Q being the quantity of electricity on the first sphere.

$$\therefore \quad C = \frac{ab}{b-a}.$$

The equations may be obtained in a simpler manner. For consider any sphere of radius r intermediate between a and b, concentric with the given spheres. Applying the theorem $\iint N ds = 4\pi M$ to this sphere, we get

$$-\frac{dV}{dr} 4\pi r^2 = 4\pi Q,$$

$$\therefore \quad \frac{dV}{dr} = -\frac{Q}{r^2},$$

$$\therefore \quad V = A + \frac{Q}{r}, \text{ or } = A + \frac{4\pi\sigma a^2}{r}.$$

We get for the value of the potential at any point between the spheres at a distance r from the centre

$$V = \frac{V_1 - V_2}{b-a} \cdot \frac{ab}{r} + \frac{bV_2 - aV_1}{b-a}.$$

(2) For the parallel planes. Take the axes of x and y in the surface of one plane at potential V_1, the other being at potential V_2. Let σ be the surface-density on the plane at potential V_2. Laplace's equation becomes

$$\frac{d^2V}{dz^2} = 0.$$

$$\therefore \quad V = A + Bz,$$

$$\therefore \quad \frac{dV}{dz} = B,$$

$$\therefore \quad B = 4\pi\sigma,$$

$$\therefore \quad V_1 = A,$$

and

$$V_2 = A + 4\pi\sigma a;$$

$$\therefore \quad V_2 - V_1 = 4\pi\sigma a,$$

and

$$Q = S\sigma.$$

$$\therefore \quad C = \frac{S}{4\pi a}.$$

The potential at any point distant z from the plane of potential V_1 is $V_1 + 4\pi\sigma z$.

(3) For the coaxal cylinders. Transforming Laplace's equation to cylindrical coordinates, and remembering that

$$\frac{dV}{d\theta} = 0, \text{ and } \frac{dV}{dz} = 0,$$

we get
$$r\frac{d^2V}{dr^2} + \frac{dV}{dr} = 0,$$

$$\therefore \quad V = A\log\frac{B}{r},$$

$$\therefore \quad \frac{dV}{dr} = -\frac{A}{r},$$

$$\therefore \quad \frac{A}{a} = 4\pi\sigma.$$

$$\therefore \quad V_1 = 4\pi\sigma a\log\frac{B}{a},$$

and
$$V_2 = 4\pi\sigma a\log\frac{B}{b}.$$

$$\therefore \quad V_1 - V_2 = 4\pi\sigma a\log\frac{b}{a}.$$

And
$$Q = 2\pi\sigma a l,$$

$$\therefore \quad C = \frac{l}{2\log\frac{b}{a}}.$$

We get for the potential at any point at a distance r from the axis

$$V_1 + 4\pi\sigma a\log\frac{a}{r},$$

$$\text{or} \quad V_2 + 4\pi\sigma a\log\frac{b}{r},$$

$$\text{or} \quad \frac{V_1\log b - V_2\log a}{\log\frac{b}{a}} - \frac{V_1 - V_2}{\log\frac{b}{a}}\log r.$$

5. Suppose we have two parallel plates at a distance h apart kept at potentials V_1 and V_2. Consider the force with which

a portion S of the first, a long way from the edges, attracts the equal portion, opposite to it, of the other. We have seen that these two portions may be considered as forming a condenser of capacity $S/4\pi h$. So that the energy of the charge, as far as these two portions are concerned, is

$$\frac{S}{8\pi h}(V_1 - V_2)^2.$$

Thus the force on S tending to increase h is

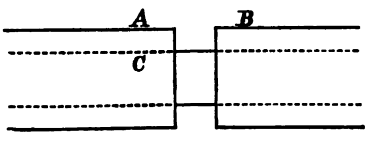
$$-\frac{S}{8\pi h^2}(V_1 - V_2)^2. \quad \text{[See Chap. III, § 22.]}$$

Or the force of attraction on S is

$$\frac{S}{8\pi h^2}(V_1 - V_2)^2.$$

It may be seen that this expression agrees with what we would obtain by considering the stress per unit area on S; for this is $2\pi\sigma^2$, and $\sigma = -(V_1 - V_2)/4\pi h$.

6. As another example, suppose we have two coaxal hollow cylinders, A, B, of equal radii, and separated by a short interval, and a third smaller cylinder C placed inside and coaxal with these.



Let us suppose all the cylinders so long that we may consider the densities on them near the interval between A and B to be always the same as they would be on the supposition that the cylinders were of infinite length. Let the three be kept at potentials V_1, V_2, V_3.

To find the force with which C is urged along its axis from A to B.

If C moves along its axis from A to B through a distance x, let us consider how the energy of the system is altered. The distribution at all points near the interval between A and B will remain the same as it was before with reference to A and B, but will rearrange itself with reference to C as C moves.

And the distribution near the ends of C will remain the same with reference to C as it was before. Thus the effect of the motion is merely to alter the lengths of those parts of the cylinders on which the distribution is uniform and the same as that on infinite cylinders.

Let a be the capacity per unit length of a condenser composed of C and either A or B.

Then the ~~in~~crease of the energy W is de ?

$$\tfrac{1}{2}ax(V_3 - V_2)^2 - \tfrac{1}{2}ax(V_3 - V_1)^2 = ax(V_1 - V_2)\left(V_3 - \frac{V_1 + V_2}{2}\right).$$

And the required force is

$$\frac{dW}{dx} = a(V_1 - V_2)\left(V_3 - \frac{V_1 + V_2}{2}\right).$$

CHAPTER V.

Specific Inductive Capacity.

1. If instead of using air as the dielectric in a condenser, that is in the space through which the induction takes place, we replace it by some other dielectric, it is found that the capacity of the condenser is always changed in a certain ratio (increased for all solid and liquid dielectrics) which depends only on the nature of the new dielectric used.

Def.: **The specific inductive capacity of a substance is the ratio of the capacity of a condenser having the given substance as dielectric to the capacity of the same condenser having air as dielectric.**

Strictly speaking, induction takes place, and lines of force pass, in the whole space surrounding the condenser, but practically it is sufficient to have only a very limited space filled up with the dielectric; for very nearly all the lines of force passing from one conductor to the other will be contained in a very limited space, although theoretically they occupy the whole of the space about the condenser.

2. Whatever be the form of a condenser; if it has given charges, the potential difference between its conductors is, with the given dielectric, $1/K$ of what it would be with air. It follows that the potential produced at any point by the given charges is $1/K$ of what it would be in air. Hence if an electrical field is made up of any charged conductors, and in one case we imagine these immersed in air, and in another in the dielectric,

the potential, and hence also the intensity at any point in the second case, will be $1/K$ of what it is in the first case.

Thus the potential and intensity produced by given charges will depend in no way on the *nature* of the conductors, however much space these may occupy, but only on the nature of the non-conducting medium of the field.

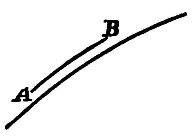
The potential produced by an element of charge e at a distance r is e/rK. And the force of repulsion between two concentrated charges q, q' is qq'/r^2K.

3. If then with a given system of conductors having given charges, instead of using air we use a dielectric of specific inductive capacity K, the law of distribution on the conductors will be the same as in air, and the lines of force and equipotential surfaces will be the same.

In the given dielectric the electrical intensity just outside a charged surface is $4\pi\sigma/K$, and the stress per unit area on the surface is $2\pi\sigma^2/K$.

4. Suppose we have a charged conductor in a medium whose specific inductive capacity, varying from point to point in any manner, is K just outside any given point on the surface of the conductor: then, if σ is the surface-density of the electricity at that point, the electrical intensity just outside that point is $4\pi\sigma/K$.

Let AB be a portion of an equipotential surface drawn just outside the surface of the conductor, at an indefinitely small distance dn from it.



Now suppose AB to become *an infinitely thin conductor* in the neutral state. This will not disturb the field. The neutral state is therefore the state of electrical distribution on AB after it becomes a conductor. Now AB is in the neutral state as far as concerns the field, that is the quantity of electricity in any indefinitely small space is zero. But what would really happen would be this. Electricity would be induced on AB, so that there would be equal and opposite densities on opposite sides of AB at every point.

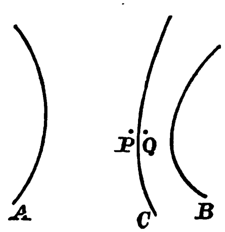
Now consider a portion of AB of area s to be taken, so small that we may make the following assumptions. That s is a plane surface: that the lines of force running from s to the Condenser given conductor all end on a portion s' of the conductor, which is also plane and equal to s: that the specific inductive capacity of the medium between s and s' is constant and equal to K.

Then the capacity of the ~~conductor~~ formed of s and s' is $\frac{Ks}{4\pi dn}$. Let dV be the excess of the potential at AB over that of the given conductor. Then $s\sigma$ being the charge on the portion s' of the conductor,

$$s\sigma = -\frac{Ks\,dV}{4\pi dn}.$$

∴ the electrical intensity just outside s' is

$$-\frac{dV}{dn} = \frac{4\pi\sigma}{K}.$$



5. Suppose we have two conductors whose surfaces are A and B, with given electrical charges on them, and contained in a practically infinite dielectric medium of specific inductive capacity K_1. Let C be any equipotential surface in the given medium. Suppose the whole of C to become an infinitely thin conductor. This does not disturb the potential at any point. Now suppose that from all the region between C and B the first dielectric is removed and replaced by a second one of specific inductive capacity K_2. This does not disturb any of the given distributions, or the forms of the equipotential surfaces or lines of force, but the potential difference between C and B becomes $\frac{K_1}{K_2}$ of what it was before. And this is true wherever the equipotential surface C is taken between A and B: so that the fall of potential along any portion of a line of force between C and B becomes $\frac{K_1}{K_2}$ of what it was in the

first dielectric. Therefore V being the potential at any point between C and B, and r a distance measured along a line of force, $\frac{dV}{dr}$ at any point becomes in the second dielectric $\frac{K_1}{K_2}$ of what it was in the first. Now if we remove the conducting sheet at C we do not disturb the field in any way.

Let us consider the relation between the electrical intensities at two indefinitely near points P and Q on opposite sides of C in the first and second dielectrics respectively. Let V_1 and V_2 be the potentials at P and Q. When the first dielectric occupied the whole space between A and B, $\frac{dV_1}{dr}$ and $\frac{dV_2}{dr}$ were equal. But now $\frac{dV_2}{dr}$ is $\frac{K_1}{K_2}$ of what it was before. Therefore we have the relation $K_2\frac{dV_2}{dr} = K_1\frac{dV_1}{dr}$.

If F_1 and F_2 are the electrical intensities at P and Q,

$$K_2F_2 = K_1F_1.$$

6. Then this is the relation we find between the electrical intensities on opposite sides of the surface of separation of the two dielectrics when their directions are at right angles to this surface of separation.

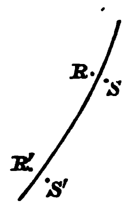
We may notice that if at a point P in the surface of separation of two media, M_1 and M_2, the direction of the line of force in M_1 is at right angles to the surface, its direction in M_2 will also be at right angles to the surface. For the equipotential surfaces in the homogeneous medium M_1 being at right angles to the lines of force, that at P must be tangential to the surface of separation. And the lines of force in M_2 cutting the equipotential surfaces at right angles, that passing through P must be at right angles to the surface of separation at P.

7. Let P be a point on the surface of separation of two media, M_1 and M_2, of specific inductive capacities K_1 and K_2. Suppose a distribution which produces a line of force through P at right angles to the surface of separation. Let F_1 and F_2

be the electrical intensities at P in M_1 and M_2. Then we have the relation

$$K_1F_1 = K_2F_2.$$

Now let us remove this distribution, and let us suppose one which produces a line of force at P parallel to the surface of separation. Now the potentials V_1 and V_2 at any two indefinitely near points R and S, just inside M_1 and M_2, must be equal. And those at two other indefinitely near points R' and S' just inside M_1 and M_2 must also be equal. So that r being now measured along one of the present lines of force, we get

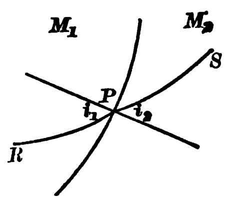


$$\frac{dV_1}{dr} = \frac{dV_2}{dr}.$$

Thus the electrical intensities just inside the two media are equal, and two lines of force indefinitely close to each other and just inside the two media are parallel.

Now let us superpose these two distributions. Each of them will produce a component of electrical intensity at any point independently of the other.

Thus any distribution giving rise to a line of force such as RPS passing through a surface of separation between two media, M_1 and M_2, may be considered to be made up of two distributions, one of which produces an electrical intensity at P in each medium at right angles to the surface of separation (for we have seen that if the intensity in one medium at P is at right angles to the surface of separation, that in the other medium at P is at right angles to the same surface), and the other of which produces intensities at P just inside the two media equal to each other and parallel to each other, and to the surface of separation.



Now let the components of the intensity at P in M_1 be F_1

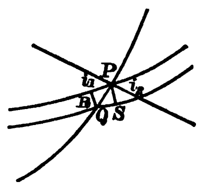
and F at right angles and parallel to the surface of separation, and those at P in M_2 be F_2 and F in the same directions. We see that F_1 and F_2 may be supposed to be produced by one distribution and the components F by another. Let i_1 and i_2 be the angles which the lines RP and PS make at P with the normal to the surface of separation.

Then we have
$$K_1F_1 = K_2F_2;$$
$$\therefore \quad \frac{1}{K_1}\frac{F}{F_1} = \frac{1}{K_2}\cdot\frac{F}{F_2},$$
that is
$$\frac{1}{K_1}\tan i_1 = \frac{1}{K_2}\tan i_2.$$

Also since the component F is in the same direction just inside each medium, the directions of RP and PS at P are in a plane with the normal to the surface of separation.

These two results may be called the laws of refraction of a line of force from one medium into another.

8. Consider a small tube of force passing across the surface of separation of the two media M_1 and M_2 and intercepting on it a piece of area s. Let F_1 and F_2 be the electrical intensities just inside M_1 and M_2, i_1 and i_2 the angles their directions at the surface of separation make with the normal.



Then
$$K_1F_1\cos i_1 = K_2F_2\cos i_2.$$

Now the areas of the cross-section of the tube just inside M_1 and M_2, which we may call s_1 and s_2, are $s\cos i_1$ and $s\cos i_2$.
$$\therefore \quad K_1F_1s_1 = K_2F_2s_2.$$

We see thus that as we pass along an elementary tube of force, K, F, and s denoting the specific inductive capacity, electrical intensity, and cross-section of the tube at any point, the quantity KFs remains constant, the tube being supposed to contain no electricity.

Now consider a tube of any size, and by dividing it up into

an infinite number of elementary tubes of force, and applying this result to each of them, we get the result that $\iint KF\,ds$ is constant throughout the tube.

9. Consider an elementary tube of force having one end on the surface of a charged body, the surface-density being σ at that point, and F being measured along the tube *from* this surface. Just outside the surface $KF = 4\pi\sigma$. If e is the quantity of electricity on the part of the surface forming the end of the tube, KFs at the surface $= 4\pi e$. $\therefore$ the value of KFs at any point of the tube is $4\pi e$.

If the other end of the tube is on a surface having a charge e', then KFs for any point of the tube is equal to $-4\pi e'$, because F is measured *to* the surface with the charge e'. Therefore $e = -e'$.

We have then the conclusion that every tube of force begins and ends on equal and opposite quantities of electricity, the tube containing no electricity inside it.

10. If any closed surface be drawn containing a quantity Q of electricity inside it, and if N be the normal component of the electrical intensity at any point of the surface, and K the specific inductive capacity at that point, then taking the integral all over the surface we have

$$\iint KN\,ds = 4\pi Q.$$

For if any elementary tube of force has an end inside the surface on a quantity e of electricity, and intercepts on the surface an elementary area ds making an angle i with the normal to ds, and F is the electrical intensity at ds measured outwards, then

$$KF\,ds\cos i = 4\pi e.$$

That is,

$$KN\,ds = 4\pi e.$$

Thus taking every elementary portion of the surface and the tube crossing it, we get $\iint KN\,ds = 4\pi Q$.

11. DEF.: **The product KF at any point is called the INDUCTION at that point.**

In air, for which K is always unity, the induction is equal to the electrical intensity.

The result, $\iint KN ds = 4\pi Q$, may be expressed as follows:

The surface-integral of normal induction over a closed surface is equal to 4π times the entire quantity of electricity contained inside the surface.

This is a generalization of the proposition already proved for air as the dielectric medium, and expressed as

$$\iint N ds = 4\pi Q.$$

The quantity $\iint KF ds$ taken over any portion of an equipotential surface is sometimes called *the flow of induction.* And we have shown that the flow of induction along a tube of force, containing no electricity, retains a constant value throughout the tube.

Lines of force, considered as the lines along which induction takes place between electrical charges, are sometimes called lines of induction.

12. MODIFIED FORMS OF POISSON'S AND LAPLACE'S EQUATIONS.

Let us apply the proposition $\iint KN ds = 4\pi Q$ to the infinitesimal parallelopiped $dx\, dy\, dz$, the volume density at which is ρ.

The term contributed by the integral for the side $dy\, dz$ is

$$K\frac{dV}{dx}\, dy\, dz.$$

That for the opposite side is

$$-\left[K\frac{dV}{dx} + \frac{d}{dx}\left(K\frac{dV}{dx}\right)dx\right] dy\, dz.$$

These two give rise to

$$-\frac{d}{dx}\left(K\frac{dV}{dx}\right) dx\, dy\, dz.$$

We get similar expressions from the other two pairs of sides.

Adding them up and putting the result equal to $4\pi\rho\,dx\,dy\,dz$, we get

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right)+\frac{d}{dy}\left(K\frac{dV}{dy}\right)+\frac{d}{dz}\left(K\frac{dV}{dz}\right)=-4\pi\rho.$$

This is what Poisson's equation becomes in a medium whose specific inductive capacity varies in any manner.

Putting $\rho = 0$ we get Laplace's equation for such a medium.

13. Let σ be the surface-density at any point on the surface of separation of two media of specific inductive capacities K_1, K_2. Let V denote the potential at any point, and n_1, n_2 the normals drawn from the given point into the two media. Construct an indefinitely small cylinder having its faces parallel to the surface of separation at the given point, and each of area s, and its sides indefinitely small as compared with its faces. Take the surface-integral of normal induction over this cylinder. Since it contains a quantity $s\,\sigma$ of electricity, and the part of the integral arising from the sides is indefinitely small compared with that arising from the faces, we have

$$K_1\frac{dV}{dn_1}\cdot s+K_2\frac{dV}{dn_2}\cdot s+4\pi s\sigma=0.$$

Thus
$$K_1\frac{dV}{dn_1}+K_2\frac{dV}{dn_2}+4\pi\sigma=0.$$

At the surface of separation there is an *apparent electrification* arising from the electrical intensities in the neighbourhood. Let the surface-density of apparent electrification be σ'. This is the surface-density with which the given electrical intensities may exist when each of the specific inductive capacities is unity.

Thus we have the equation

$$\frac{dV}{dn_1}+\frac{dV}{dn_2}+4\pi\sigma'=0.$$

14. Distribution of Electric Energy.

Consider two elements of charge e and $-e$, at the ends of a tube of induction, and at potentials V and V'. The energy arising from these may be considered as $e\,(V-V')\,/\,2$. Now

Maxwell considered that all the energy of the field resides in the non-conducting medium between the conductor; and that the above amount resides in the tube connecting e and $-e$.

To consider how this energy is distributed along the tube we may reason as follows. We may suppose the tube terminated by a charge $-e$ placed where the potential is V', without altering the potentials, or state of the medium, between this and the charge e. The energy of this tube is $e\,(V - V')\,/\,2$. This is therefore the energy of the part of the whole tube considered between potentials V and V'. Thus we must suppose the energy distributed throughout the tube so that there is between any two equipotential surfaces an amount proportional to the difference of their potentials.

Consider a portion of the tube of cross-section s, and length dn, where the intensity is I, and specific inductive capacity K. The energy in this portion is

$$-\tfrac{1}{2}e\frac{dV}{dn}\,dn = \tfrac{1}{2}e\,I\,dn.$$

But $$4\pi e = sKI.$$

Thus energy in part considered is

$$\frac{K}{8\pi}\,I^2.\,s\,dn.$$

Or the energy for unit volume, at any point, is

$$\frac{KI^2}{8\pi}.$$

15. Maxwell's Theory of Electric Displacement.

According to this view Maxwell supposes that there is across any section of this tube of induction what he calls an *electric displacement* of amount e, the displacement in an isotropic medium being always in the same direction as the intensity. We may regard the intensity I, and the displacement per unit area (which is e/s, or $KI/4\pi$), as analogous to stress and strain in an elastic solid.

Then for the energy per unit volume, given by the formula—*stress* $\times$ *strain* $\div 2$, we get, as before,

$$\tfrac{1}{2} \times I \times KI/4\pi = \frac{KI^2}{8\pi}.$$

Again, the quotient—*intensity÷displacement*, corresponding to *stress÷strain*, or *elasticity* in the solid, and which has the value $K/4\pi$, has been called the *electric elasticity* of the medium.

16. We have noticed that the amount of electric displacement is the same throughout the tube of induction, however the nature of the medium may vary from point to point of the tube; and that it is equal to the quantity of electricity from which the tube starts, and equal to the quantity, with sign changed, on which it stops. We must suppose that charge on a body means displacement in the medium surrounding the body, the displacement beginning (or ending) where the charge is, and continuing along the corresponding lines of induction, being a displacement outwards from + charged surfaces, and inwards to − charged surfaces.

It follows that if any closed surface be drawn containing only parts of the non-conducting medium, by introducing charges on the conductors of the field the amounts of displacement produced into and out of this surface are equal.

Again, if a closed surface contains a conductor, or any part of a conductor, on introducing + charge into it, on the conductor, an amount of displacement, equal to this charge, takes place out of this surface along the lines of induction which cut it and start from the charge; and on introducing − charge into it an amount of displacement, equal numerically to this charge, takes place into the surface along the lines of induction which cut it and end on the charge.

Now, regarding what has been called electric displacement across any surface as a movement of so much + electricity in the + direction of the lines of induction, it follows from what has just been said that no change can be produced in the entire quantity of electricity inside any closed surface, drawn in any manner in a field; the amounts that pass in and pass out are numerically equal. If, for example, a closed surface encloses a portion of a surface of a conductor which receives a negative charge, this

introduces −charge into the closed surface, but at the same time an equal amount of electric displacement, along the lines of induction ending on this −charge, takes place into the closed surface.

On this account electricity is said to behave as an incompressible fluid.

CHAPTER VI.

Electrical Images and Inversion.

1. **ELECTRICAL IMAGES.** The method of electrical images is a method given by Sir William Thomson for investigating electrical distributions in certain cases.

Def.: **AN ELECTRICAL IMAGE is an electrified point or system of points on one side of a surface which would produce on the other side of that surface the same electrical action which the actual electrification of that surface really does produce.**

2. The theory of electrical images may be made to depend on the following proposition.

The potential of a surface being given and the electrical distribution on one side of it, then the potential and electrical intensity at every point on that side is fixed.

Let V and V' be possible values of the potential of any point P with the given potential of the surface, and the given distribution on one side of it, P being on this side.

Let us superpose the distribution producing potential V at P, and that producing potential V' at P reversed. Then we have all the surface at zero potential, and no charges in the field on the side of it we are considering. Thus there can be no lines of force in the field, and every point in the field is at zero potential. Therefore $V = V'$.

Thus the potential is fixed for every point of the field, and therefore the direction and magnitude of the electrical intensity is fixed for every point in the field.

3. To apply this theorem to electrical images.

Suppose for any electrical distribution we draw an equipotential surface at potential V, and determine the electrical intensity F at every point on this surface. Then, if we keep this surface at potential V and keep all the charges on one side of it as they were before, the intensity at any point just outside the surface, and on this side, remains what it was before.

Let us then introduce a conductor into the field having its surface coinciding with the given equipotential surface, and keep it at potential V. The intensity at any point just outside it is F, the same as that at the same point due to the original distributions on both sides of the equipotential surface. Having then determined F for any point on the surface due to the given distributions, the surface-density at that point of the conductor is $F/4\pi$.

If then we have a conducting surface at potential V under the action of any distribution on one side of it, we can find the density at any point of it if we can determine a distribution on the other side of the surface, which with the given one has the given surface for an equipotential at potential V, and if we can calculate the electrical intensity at all points of the equipotential for the two distributions.

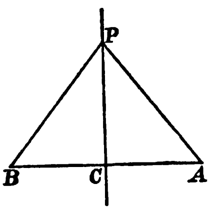
4. Call the original distribution A, and its image with reference to the surface at potential V, B. We may plainly regard A as the image of B with reference to this surface. For if we introduce a conductor with a surface coinciding with this surface and facing B, and keep it at potential V, we may then remove A, and the potential and electrical intensity will be the same at every point on the side of B as they were before.

The electrical density at any point of this surface is equal and opposite to that which was at the corresponding point of the other conductor whose surface faced A. For the intensity F is the same just on one side of the equipotential surface as it is at the other at the same point, and in one case it is directed to the surface, and in the other away from it.

5. The entire quantity of electricity on the surface of a conductor under the influence of a distribution A is equal to the entire quantity in the image of A with reference to the surface, the distribution A being outside the surface.

Suppose an infinitely thin conducting sheet put to coincide with the surface; A and its image being only in the field. And let Q be the entire quantity of electricity in the image. A quantity $-Q$ of electricity will be induced on the inner surface of the sheet, and there will be a quantity Q on its outer surface. But this will be the quantity on any conducting surface facing A and coinciding with the given surface and kept at the given potential.

6. To find the surface-density at any point of an infinite plane kept at potential zero, under the action of a charge e concentrated at a point at a distance a from the plane.



Let the charge e be at A. Draw AC perpendicular to the plane, and produce it to B, making CB equal to AC.

The charge e at A with a charge $-e$ at B will maintain a surface coinciding with the given plane at potential zero. The charge $-e$ at B is then the image of e at A with reference to the given plane.

Let P be any point on the plane. The electrical intensity at P due to e and $-e$ is normal to the plane and, measured towards the side on which A lies, its value is

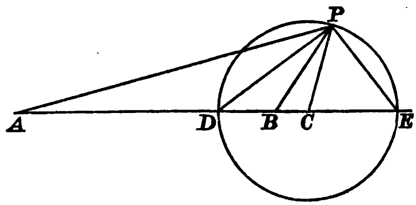
$$-2\frac{e}{AP^2}\cdot\frac{AC}{AP}=-\frac{2ea}{AP^3}.$$

Therefore the surface-density at the point P of the plane at potential zero, under the action of e at A, is

$$-\frac{ea}{2\pi AP^3}.$$

7. To find the surface-density at any point of a sphere of radius a kept at potential zero, under the action of

a charge e concentrated at a point at a distance b from its centre.



Let e be at the point A; and C the centre of the sphere.

Now suppose we have two charges e, and $-e'$ at two points A and B. The surface of zero potential determined by them is given by the equation

$$\frac{e}{r}=\frac{e'}{r'}.$$

P being any point of this surface, we have

$$\frac{AP}{BP}=\frac{e}{e'}.$$

The internal and external bisectors of the angle APB meet AB in two fixed points D and E, such that

$$\frac{AD}{DB}=\frac{AE}{BE}=\frac{e}{e'}.$$

Remembering that DPE is a right angle, we see that the locus of P is the sphere having DE as a diameter.

Also, we can easily show that

$$\frac{e}{e'}=\frac{CA}{CD}, \text{ and } CB\,.\,CA=CD^2.$$

Thus the image of e at A with reference to the given sphere is $-e\,.\,\frac{a}{b}$ at a point B, such that $CB=\frac{a^2}{b}$.

We have to determine the electrical intensity at any point P of the sphere under the action of e at A and $-e\,.\,\frac{a}{b}$ at B.

This consists of two components, $\frac{e}{AP^2}$ along AP, and $-\frac{e}{BP^2}\cdot\frac{a}{b}$ along BP.

Resolving these in the directions AC and CP, we get

$$\frac{e.b}{AP^3} - \frac{e.BC}{BP^3} \cdot \frac{a}{b} \text{ in the direction } AC,$$

$$\frac{ea}{AP^3} - \frac{e}{BP^3} \cdot \frac{a^2}{b} \text{ in the direction } CP.$$

Now the triangles CBP, CPA are similar.

Thus $$BP^3 = AP^3 . \frac{a^3}{b^3},$$

and $$BC = \frac{a^2}{b}.$$

Therefore the component of the intensity at P in the direction AC vanishes, as it should do, this intensity being entirely in the direction CP.

We have for its value

$$ea\left(\frac{1}{AP^3} - \frac{1}{BP^3} \cdot \frac{a}{b}\right)$$

$$= \frac{ea}{AP^3}\left(1 - \frac{b^2}{a^2}\right)$$

$$= - e\frac{b^2 - a^2}{aAP^3}.$$

Therefore the surface-density at P is

$$- \frac{e(b^2 - a^2)}{4\pi a AP^3}.$$

If the charge e at A is inside the spherical hollow DPE, its image at B is outside.

We would find now for the surface-density induced inside at P the value

$$- \frac{e(a^2 - b^2)}{4\pi a AP^3}.$$

In both cases we get for the density at P the value

$$- \frac{e.AD.AE}{4\pi CP} \cdot \frac{1}{AP^3}.$$

The surface-density at any point due to any number of

concentrated charges is found by superposing those due to each charge separately.

8. Suppose the sphere to be at potential V. This would be maintained by a uniform surface-density

$$\frac{V}{4\pi a},$$

with no other charges in the field.

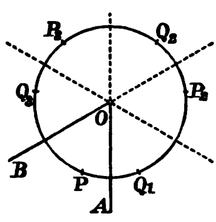
Thus the surface-density at potential V is found by superposing this and the surface-densities due to the various charges.

The entire quantity of electricity on the sphere at potential V under the action of charges e_1, e_2, e_3,...at distances b_1, b_2, b_3 from the centre, is equal to the charge which would alone keep it at potential V together with all the charges in the images; that is, to

$$a\left(V - \frac{e_1}{b_1} - \frac{e_2}{b_2} - \frac{e_3}{b_3} - \ldots\right).$$

9. Suppose we have two planes intersecting at an angle $\frac{\pi}{n}$, where n is any integer; and at potential zero, under the action of a charge e at any point P between them. We can find the surface distributions of the planes by means of electrical images.

Let AO, BO be the traces of the planes on the plane passing through P and at right angles to each of them. Let this plane cut the line of intersection of the given planes in O. Describe a circle with O as centre and OP as radius.



Let $\angle AOP = a$, and take a series of points Q_1, P_2, Q_2, &c. round the circumference of the circle, such that

$$\angle AOQ_1 = a,$$

$$\angle AOP_2 = \frac{2\pi}{n} - a,$$

$$\angle AOQ_2 = \frac{2\pi}{n} + a,$$

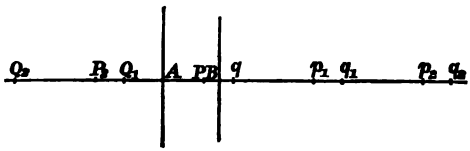
and so on.

Now suppose we have a charge e at each of the points P_2, P_3, &c., and a charge $-e$ at each of the points Q_1, Q_2, Q_3, &c. These charges will clearly keep the planes OA, OB at potential zero. Thus they are the image of the charge e at P with respect to the given planes.

We can then by determining the electrical intensity at any point of the given planes due to the charges at the points P, Q_1, P_2, Q_2, &c., find the density of the electrical distribution at any point.

10. INFINITE SUCCESSION OF IMAGES.

Suppose we have two parallel planes, of indefinite extent at potential zero, under the action of a charge e at a point P between them. We can find the distribution at any point on either of them by electrical images.



Let the straight line through P perpendicular to the two planes meet them in the points A and B. And let $PA = a$, $PB = b$, and $a + b = c$.

Take an infinite series of points, Q_1, P_2, Q_2, &c., to the left of A on the straight line AB, such that

$$AQ_1 = a,$$
$$AP_2 = 2c - a,$$
$$AQ_2 = 2c + a,$$

and so on.

And take an infinite series of points, q, p_1, q_1, p_2, q_2, &c., to the right of B, such that

$$Bq = b,$$
$$Bp_1 = 2c - b,$$
$$Bq_1 = 2c + b,$$

$$Bp_2 = 4c - b,$$
$$Bq_2 = 4c + b.$$

Now if we suppose a charge e placed at each of the points $P_2, P_3, \ldots, p_1, p_2, \ldots$, and a charge $-e$ placed at each of the points $Q_1, Q_2, \ldots, q, q_1, q_2, \ldots$, these with the charge e at P will keep each of the planes at potential zero.

This infinite system of charges is then the image of the charge P with respect to the two planes.

The electrical intensity at any point on the plane through A, at a distance r from A, due to two charges e and $-e$, at two points on the line AB at distances l to the right and left of A is normal to this plane and equal to

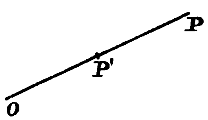
$$-\frac{2el}{\{l^2+r^2\}^{\frac{3}{2}}}.$$

Thus the density at any point on the plane through A at a distance r from A is equal to

$$-\frac{e}{2\pi}\sum_0^\infty \frac{(2n+1)c-b}{[\{(2n+1)c-b\}^2+r^2]^{\frac{3}{2}}}$$
$$+\frac{e}{2\pi}\sum_0^\infty \frac{(2n+1)c+b}{[\{(2n+1)c+b\}^2+r^2]^{\frac{3}{2}}}.$$

And we find a similar expression for the density at any point on the plane passing through B.

11. GEOMETRICAL INVERSION. Let O be a fixed point. Join O to any point P, and take the point P' on OP such that



$$OP \cdot OP' = a^2.$$

Then P' is called the inverse of the point P with respect to a sphere of centre O and radius a. And the process of obtaining P' in this way from P is called inverting P with respect to the sphere. O is called the centre of inversion.

If we invert every point on a given surface, or within a given volume with reference to a given sphere, it is obvious that the inverse obtained will be a new surface or a new volume.

12. The following propositions in Geometrical Inversion will be found useful.

(1) Every circle in a plane with the centre of inversion becomes, when inverted, a circle, except it passes through the centre of inversion, and then it becomes a straight line.

For if b is the distance of the centre of the given circle from O, and c its radius, its equation is, taking the origin of polar coordinates at O, and the initial line passing through the centre of the given circle,

$$\rho^2 - 2b\rho\cos\theta + b^2 - c^2 = 0.$$

For the inverse we must put $\frac{a^2}{\rho}$ for ρ.

Thus we get

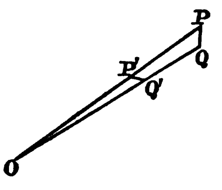
$$\rho^2 - \frac{2a^2b}{b^2-c^2}\rho\cos\theta + \frac{a^4}{b^2-c^2} = 0.$$

This shows that the inverse is a circle, unless $b = c$, in which case it becomes the straight line

$$2b\rho\cos\theta - a^2 = 0.$$

(2) From this it follows that the inverse of a sphere is a sphere unless it passes through the centre of inversion, and then it becomes a plane.

13. (3) The angle between two lines is equal to the angle between their inverses.



For P, Q being two points, and P', Q' their inverses, the triangles OPQ, $OQ'P'$ are similar, and the angle OPQ is equal to the angle $OQ'P'$. Therefore in the limit when P and Q are indefinitely near together, we get $\angle OPQ = \angle Q'P'P$.

Therefore two lines making the same angles with $OP'P$ as their inverses do, the two lines will cut at the same angle as their inverses.

(4) From this it follows that two surfaces cut each other at the same angle as their inverses.

14. ELECTRICAL INVERSION. Suppose any group of volumes and surfaces in which there is any given electrical distribution to be inverted with respect to a sphere of radius a. And let every indefinitely small portion of electricity be replaced in the inverse by its image with respect to the given sphere.

Let e and e' denote indefinitely small corresponding charges, either surface- or volume-distributions, at two corresponding points P and P', distant r and r' from O.

Then
$$\frac{e'}{e} = \frac{a}{r} = \frac{r'}{a}.$$

Let s, s' denote two indefinitely small corresponding areas, and v, v' two indefinitely small corresponding volumes at P and P'.

Then
$$\frac{s'}{s} = \frac{a^4}{r^4} = \frac{r'^4}{a^4},$$

and
$$\frac{v'}{v} = \frac{a^6}{r^6} = \frac{r'^6}{a^6}.$$

Let σ, σ' denote the surface-densities, and ρ, ρ' the volume-densities at P, P'.

Then combining the previous results, since

$$\sigma' : \sigma = \frac{e'}{s'} : \frac{e}{s},$$

and
$$\rho' : \rho = \frac{e'}{v'} : \frac{e}{v},$$

we get
$$\frac{\sigma'}{\sigma} = \frac{r^3}{a^3} = \frac{a^3}{r'^3},$$

and
$$\frac{\rho'}{\rho} = \frac{r^5}{a^5} = \frac{a^5}{r'^5}.$$

Let e, e' be two corresponding charges at P and P', and let the potentials due to these at Q, Q' be V, V'.

Then
$$V = \frac{e}{PQ}, \text{ and } V' = \frac{e'}{P'Q'};$$

$$\therefore \quad \frac{V'}{V} = \frac{e'}{e} \cdot \frac{PQ}{P'Q'}$$
$$= \frac{a}{OP} \cdot \frac{OP}{OQ'};$$
$$\therefore \quad V' = \frac{Va}{OQ'}.$$

In the same way if we take a system with any electrical distribution producing potential V at a point Q, and invert it; since the entire potential produced at any point is the sum of the separate potentials produced by the various elementary charges, the potential at Q', the inverse of Q, will be

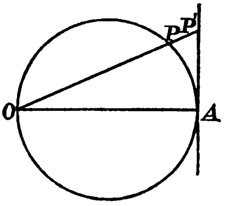
$$\frac{Va}{OQ'}.$$

15. If now we suppose the locus of all points such as Q' to be a conducting surface, and we place at O a charge $-Va$, this surface will be in electrical equilibrium at potential zero, and we can calculate the density at every point of it if we know the density at all points of the corresponding original surface.

Conversely, if we have a conducting surface at potential zero under the action of a charge e concentrated at a point O, we can get from it, by inversion with respect to a sphere of radius a and O as centre, a conducting surface at potential $-e/a$, and find the electrical distribution on it.

16. As an example, suppose a sphere of radius a raised to potential V; and let us invert it with respect to a sphere having its centre O on the surface of the given sphere, and of radius $2a$.

We get for the inverse a plane touching the given sphere at a point A, diametrically opposite to O.



The density at any point P of the sphere is

$$\frac{V}{4\pi a}.$$

Therefore that at the corresponding point P' on the plane is

$$\frac{V}{4\pi a} \cdot \frac{(2a)^3}{OP'^3} = \frac{V.OA^2}{2\pi OP'^3}.$$

Thus the plane would have this electrical density on it, and be at potential zero under the action of a charge $-2aV$ at O.

Or with a charge e at O and the plane at potential zero, the density at P' is

$$-\frac{e\,OA}{2\pi OP'^3},$$

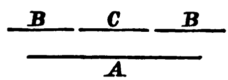
a result already obtained.

CHAPTER VII.

ELECTROMETERS.

1. An Electrometer is an instrument for measuring differences of electrical potential by means of the force exerted between bodies raised to different potentials. We shall merely consider the principles of the action of two electrometers, omitting all details about their construction and use.

2. Sir William Thomson's Attracted-Disc Electrometer and Absolute Electrometer, which works on the same principle, have the following for their essential parts:—



B is a horizontal circular plate, having a concentric circular aperture, which is very nearly filled up with a disc C; the latter being in position when its lower surface and that of B are in the same plane. A is a circular plate parallel to B and C, and whose distance from them may be adjusted by means of a micrometer screw below A.

C is suspended by metallic wires, or in the Absolute Electrometer by an arrangement of springs, and to bring it into position it must be pulled down by a force F, which is determined in dynes. It is observed whether C is in position by a suitable arrangement of lenses. B and C are kept in electrical connexion, and so at the same potential.

Let S be the area of the disc C, d the distance between C and A, V the difference of potentials between C and A.

Then $$V = d\sqrt{\frac{8\pi F}{S}}.$$

By keeping A and B at the potential difference which we wish to measure and measuring d, this potential difference may be found in this way. This is called by Sir William Thomson the 'idiostatic' method of using the instrument.

As the absolute distance between the plates is not an easy thing to measure, the instrument is generally used in the following way, called by Sir William Thomson the 'heterostatic' method of using it.

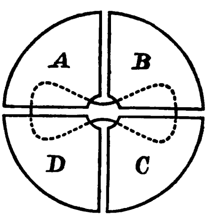
B and C are kept at a constant potential, and A is connected successively with the two points whose potential difference we wish to measure. Let V, V' be the potentials of the two points; d and d' the distances of A from B for these two connexions when C is in position.

Then $$V'-V=(d'-d)\sqrt{\frac{8\pi F}{S}}.$$

Now $d'-d$ is a quantity that can be obtained with much greater accuracy than d; for it is simply the difference of the readings of the micrometer screw for the two positions of A.

3. Sir William Thomson's Quadrant Electrometer has the following for its essential parts:—

A flat cylindrical brass box has small circular portions cut away round its axis, and is divided into its four quadrants, A, B, C, D, and placed with its axis vertical. A light flat aluminium needle swings in the centre of the box about an axis coinciding with that of the cylinder, and is suspended by a bifilar suspension; and is of such a shape that as it turns through any small angle θ the portion passing out of one quadrant into the next is proportional to θ, equal to $k\theta$, say: that is, for some extent, the ends of the needle must be portions of a circle having for centre the point where it is met by the axis. Opposite pairs of quadrants, A and C, and B and D, are joined across by wires, so that



A and C are always at the same potential, and likewise B and D.

The needle is kept at a constant high potential V. Let A and C be at potential V_1, and B and D at potential V_2. And suppose that the needle is deflected through an angle θ, passing out of quadrants A and C into quadrants D and B. Let us consider the value of the statical moment which maintains this deflexion. We shall suppose this angle to be small enough for the side-edges of the needle still to remain well enclosed within the quadrants within which they were in the position of equilibrium.

Let the capacity of the condenser, formed by a portion of the needle well enclosed within a quadrant and far away from the side-edge of the needle, be c per unit angle. Imagine the deflexion to undergo a slight increment $d\theta$, so that an angular portion $d\theta$ of the needle passes into each of the quadrants B and D, and an equal angular portion of A and C. The corresponding increment of the energy is

$$dW = \tfrac{1}{2} c\, d\theta\, (V - V_2)^2 - \tfrac{1}{2} c\, d\theta\, (V - V_1)^2$$
$$= c\, d\theta\, (V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right).$$

Therefore the moment of the force producing the deflexion θ is

$$\frac{dW}{d\theta} = c\,(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right).$$

Now V is kept very large as compared with V_1 or V_2. Thus we may suppose $V - (V_1 + V_2)/2$ to be constant. And for small deflexions the moment of torsion in the bifilar suspension is proportional to the deflexion. Thus we have, m being a constant depending on the instrument,

$$V_1 - V_2 = m\theta.$$

m is called the constant of the instrument, and its value may be determined by using a known potential difference for $V_1 - V_2$, and observing the corresponding deflexion.

PART II.

CHAPTER I.

MAGNETS.

1. CERTAIN bodies are found to possess the property that at any given locality on the earth's surface any such body tends to set itself so that a certain line in it has a definite direction with regard to the earth. Such a body is called a Magnet. The line which sets in a definite direction is called its Magnetic Axis. The magnetic axis may be found by setting the magnet so that it can move freely about its centre of gravity, and finding the line which always sets in a definite direction.

Magnets may be natural, such as the iron ore called lodestone; or artificial, such as pieces of steel in which the property has been produced by rubbing with other magnets, or by other methods, called methods of magnetizing.

2. It will be seen that the magnetic axis of a body, as above defined, is not a definite line, but only a definite direction fixed in the body; for if a certain line in the body takes a definite direction, then, of course, any line parallel to it does. For definiteness that line which passes through the centre of gravity of the body and sets in a definite direction may be considered as the magnetic axis.

If we suspend a magnet, so that it is capable of moving freely about a vertical axis perpendicular to its magnetic axis, the

magnetic axis will, as a rule, set in a definite azimuth; pointing, as a rule, more or less north and south. That end of the axis which points northwards is called the north pole, and that end which points southwards is called the south pole. These names are often abbreviated into *N.* pole and *S.* pole.

3. If we take a lot of magnets and determine their *N.* and *S.* poles, we shall find on presenting a pole of one to a pole of another which is suspended, that a *N.* and *S.* pole will mutually attract each other, but two *N.* poles or two *S.* poles will repel each other. We thus get the first law of magnetic action.

LAW I. Like poles repel each other, and unlike poles attract each other.

4. If a long thin straight piece of steel be magnetized in such a way that there is no magnetic action due to any point of it except its ends, it is said to be **uniformly magnetized.**

Uniform magnetization cannot be exactly obtained in any actual magnet; but by careful magnetization a close approximation to it can be obtained.

5. A magnetic field is any region in which magnetic influence is exercised. The region round about a magnet, for instance, is a magnetic field; and as we have already seen, the earth itself produces a magnetic field.

If we take a long thin uniformly magnetized bar, the magnetic action which this will produce, or the extent to which it will itself be acted on in a magnetic field, will depend on what we may call the **quantity of magnetism** in its poles. The *N.* pole of a magnet is said to have positive magnetism, and the *S.* pole negative magnetism.

Now let us take any number of such magnets and put them so that all their *N.* poles are together. They will as a rule (as we shall see later on) affect each other, the state of magnetization in each being to a certain extent altered by the presence of the others. But in a case in which we may neglect this action, as is nearly the case if the magnets are made of well-tempered steel, it will be found that the combined *N.* pole thus obtained is

acted upon at any given point in a magnetic field with a force equal to the sum of the forces with which the separate poles are acted on at the same point. Thus we conclude that the force with which a pole is acted on in a given field is proportional to the quantity of magnetism in it. Thus we see that if we have two given magnetic poles in given positions, the force with which they act on each other is proportional to the quantity of magnetism in each, when that in the other remains constant, or this force is proportional to the product of the quantities in the two poles, whatever be the unit of quantity of magnetism.

6. By using long thin bars, as uniformly magnetized as possible, with the torsion-balance, Coulomb showed that the force between two poles is inversely proportional to the square of the distance between them. Putting these results together, we get the second law of magnetic action.

LAW II. Two magnetic poles act on each other with a force proportional to the quantities of magnetism they contain, and inversely proportional to the square of the distance between them.

The quantity of magnetism in a pole is also called the **strength** of the pole.

Let two poles of strengths m and m' be at a distance r centimetres apart, and let F be the force of repulsion between them.

Then $$F = k\frac{mm'}{r^2},$$

where k is some constant.

As in the case of electrostatics, the action will also depend on the medium; but here too, unless the contrary is specified, we shall always suppose the medium to be air under standard conditions.

We have not yet fixed upon the unit of magnetism or unit strength of pole. Let us so take it that when m and m' each become a unit, and r a unit of length, F may be a unit of force. Then $k = 1$. And the above equation becomes

$$F = \frac{mm'}{r^2}.$$

And we have the following definition of the unit of magnetism.

DEF.: **A UNIT OF MAGNETISM is that quantity of magnetism which must be concentrated in an indefinitely small pole, so that when placed at a distance of one centimetre from an exactly similar pole it repels it with a force of one dyne.**

7. DEF.: **THE INTENSITY of a magnetic field at a given point is the force in dynes with which it acts on a unit magnetic pole placed at that point.**

A magnetic field is completely determined if we know the intensity at every point of it and the direction in which a positive pole would be urged at every point. A magnetic field is said to be uniform if each of these two things is the same at every point of it.

If the intensity of a field is H at a given point of it, a pole of strength m will be acted upon at that point with a force of mH dynes. For, from what we have already seen, the force with which a given magnetic system acts on a pole is proportional to the strength of the pole.

8. At a given locality of the earth's surface, the field due to the earth may be taken as practically uniform; for it is found that the direction in which this field tends to set the magnetic axis of a magnet, and the statical moment with which a magnet placed in a given direction is acted upon, both remain constant all throughout a given space of moderate dimensions.

From this it can be shown that the entire quantity of magnetism in any magnet is algebraically zero. For it can be shown by various experiments that the resultant magnetic force with which the earth acts on any magnet is zero. A piece of steel is found to weigh exactly the same before and after magnetization. A magnet hung at the end of a vertical thread does not cause it to deviate from the vertical. A magnet floated on a piece of wood in water does not move away bodily in any direction.

Thus the earth's field merely tends to set a magnet with its

magnetic axis in a certain direction by means of forces acting on it which are equivalent to a statical couple, and does not tend to move it bodily in any direction.

9. DEF.: **A MAGNETIC LINE OF FORCE is a line drawn in the field such that its direction at any point is the direction of the magnetic intensity at that point.**

These lines correspond to lines of force in the electrostatic field. In the same way we have equipotential surfaces and tubes of force in the magnetic field. And the state of a magnetic field may be completely shown by drawing its equipotential surfaces and lines of force, just as was done for the electrostatic field.

A great many propositions proved for the electrostatic field will hold without modification for the magnetic field. Thus we have the following:

The surface integral of normal magnetic intensity over any section of a magnetic tube of force containing no magnetism is constant throughout the tube.

This integral will be represented by the number of lines of force passing through the tube.

10. DEF.: **THE MAGNETIC POTENTIAL at any point of a magnetic field is the work which would be done by the magnetic forces of the field on a positive unit of magnetism as it moves from that point to an infinite distance.**

If σ denote the surface-density at any point of a field, and ρ the volume-density, and r denote the distance of a point P from an element of surface ds, and r' its distance from an element of volume dv, it may be shown, just as in the case of electrostatic potential, that the potential at P is

$$\iint \frac{\sigma}{r}\, ds + \iiint \frac{\rho}{r'}\, dv.$$

11. DEF.: **THE MAGNETIC MOMENT of a magnet is the statical couple with which it would be acted on by a uniform magnetic field of unit intensity if placed with its magnetic axis at right angles to the lines of force of the field.**

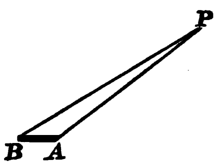
If the magnet is a bar uniformly and longitudinally mag-

netized, its magnetic moment is the product of its length into the strength of its positive pole.

Suppose a magnet of moment M to be placed in a uniform field of intensity H, so that the axis of the magnet makes an angle θ with the lines of force of the field.

We may suppose the field made up of two fields of intensities, $H\cos\theta$ and $H\sin\theta$, having their lines of force respectively parallel and at right angles to the axis of the magnet. The first of these gives rise to no statical moment acting on the magnet. So that the statical moment with which the magnet is acted on in the given field is that due to the second part, or is $MH\sin\theta$.

12. To find the potential due to an indefinitely small magnet of magnetic moment M at a point distant r from it; the distance r making an angle ϵ with the positive direction of the magnetic axis.



First, let the magnet AB be uniformly magnetized, having a positive pole m at A, and a negative pole $-m$ at B. PBA is the angle ϵ.

Potential at P is

$$m\left(\frac{1}{PA}-\frac{1}{PB}\right)$$

$$= m\frac{PB-PA}{PA.PB}$$

$$= \frac{mAB\cos\epsilon}{r^2}, \text{ in the limit,}$$

$$= \frac{M\cos\epsilon}{r^2}.$$

Next, suppose the magnet to be magnetized in any manner. We may then, by putting each little element of positive magnetism in it with an element of negative magnetism, suppose it to be made up of uniformly magnetized magnets of moments M_1, M_2,... whose axes make angles ϵ_1, ϵ_2, ... with the distance r.

Then the potential at P is, as before,

$$\frac{M_1 \cos \epsilon_1 + M_2 \cos \epsilon_2 + \ldots}{r^2}.$$

But $M_1 \cos \epsilon_1 + M_2 \cos \epsilon_2 + \ldots = M \cos \epsilon$, for each of these is the statical moment with which the magnet would be acted on by a field of unit intensity whose lines of force are at right angles to r.

Thus the potential at P is

$$\frac{M \cos \epsilon}{r^2}.$$

13. DEF.: **THE INTENSITY OF MAGNETIZATION of a magnetized body at a given point is its magnetic moment per unit volume at that point.**

That is, if dv is an element of volume of the magnetized body at a point P, and dM the magnetic moment of this volume, the limiting value of dM/dv when dv is indefinitely small is the intensity of magnetization at P.

The magnetization at any point of a magnet may be defined by its intensity and its direction. The direction is the direction of the magnetic axis, reckoned from *S.* pole to *N.* pole, of the element of volume dv.

We shall denote the intensity by I. And the direction may be defined by its direction-cosines λ, μ, ν, referred to three rectangular axes.

The magnetization at any point may also be given in terms of its three components referred to rectangular axes. Let these be A, B, C.

Thus $A = I\lambda$, $B = I\mu$, $C = I\nu$.

14. Let us consider the potential at a point P (ξ, η, ζ) due to a given magnetic system. Let the components of magnetization at any point (x, y, z) be A, B, C. Let r be the distance from (x, y, z) to the point P; and ϵ the angle between r and the direction of magnetization.

Then $r \cos \epsilon = \lambda(\xi - x) + \mu(\eta - y) + \nu(\zeta - z)$.

Thus the potential at P due to $dx\,dy\,dz$ is

$$\frac{I\cos\epsilon}{r^2}\,dx\,dy\,dz = \{A(\xi-x)+B(\eta-y)+C(\zeta-z)\}\frac{1}{r^3}\,dx\,dy\,dz.$$

Thus the entire potential at P is

$$V=\iiint\{A(\xi-x)+B(\eta-y)+C(\zeta-z)\}\frac{1}{r^3}\,dx\,dy\,dz.$$

But since $r^2=(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2$,

$$\frac{d}{dx}\cdot\frac{1}{r}=\frac{\xi-x}{r^3}, \text{ and so on.}$$

So that integrating by parts, we get

$$V=\iint A\frac{1}{r}\,dy\,dz+\iint B\frac{1}{r}\,dz\,dx+\iint C\frac{1}{r}\,dx\,dy,$$
$$-\iiint\frac{1}{r}\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)dx\,dy\,dz.$$

Or, if l, m, n are the direction-cosines of the normal to an element of surface dS, and dv is an element of volume, we get

$$V=\iint(lA+mB+nC)\frac{1}{r}\,dS$$
$$-\iiint\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)\frac{1}{r}\,dv.$$

Thus we may suppose the given magnetization to be produced by a surface-density σ, and a volume-density ρ, such that

$$\sigma = lA+mB+nC,$$

and $$\rho=-\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right).$$

15. Suppose we have an indefinitely short magnet of length ds, and pole strength m, placed in a field so that its negative and positive poles are at points whose potentials are V and V'. Then the potential of the magnet in the field is

$$m(V'-V)=m\frac{dV}{ds}ds=M\frac{dV}{ds},$$

where M is the moment of the magnet, and s is measured along the positive direction of its axis.

If H is the strength of the field at the magnet, and ϕ the angle between the positive directions of the field and the magnet's axis, the potential becomes

$$-MH\cos\phi.$$

Thus, if at a point in a magnetic substance, the field strength is H, and the intensity of magnetization I, the energy, per unit volume, of the given state of magnetization is $-IH\cos\phi$, ϕ being the angle between the directions of the intensity, and of the magnetization at the given point, and H being supposed to remain constant during the magnetization.

If H varies with I, the small increase of energy per unit volume, during increase dI in the magnetization, in which H may be supposed constant, is $-H\,dI\cos\phi$. And the entire energy of magnetization, per unit volume, is

$$-\int H\cos\phi\,dI.$$

This expression denotes the work done *against the field* in producing the magnetization, or that which the field would do in destroying it. The work done, per unit volume, *by the field* during the magnetization is, of course,

$$\int H\cos\phi\,dI.$$

16. DEF.: **A MAGNETIC SOLENOID is an indefinitely narrow filament magnetized everywhere in the direction of its length.**

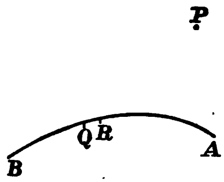
DEF.: **THE STRENGTH of a solenoid at any point is the product of the intensity of magnetization and the area of the cross-section at that point.**

A solenoid is called *Simple* or *Complex* according as its strength is the same throughout, or varies from point to point.

Let a be the area of the cross-section at a given point of a solenoid, M the magnetic moment of an element of length ds at this point, I the intensity, and m the strength at this point.

$$\text{Then}\quad m = Ia,$$
$$\text{and}\quad Ia\,ds = M,$$
$$\therefore\quad M = m\,ds.$$

Suppose we have a simple solenoid AB, BA being the direction of magnetization; m the strength. Let P be a point whose distances from the ends A and B are r_1 and r_2.



Let QR be an element of length ds in the solenoid, such that $PQ = r$, $PR = r + dr$. Then the potential at P due to QR is

$$\frac{m\,ds}{r^2}\cos PQR = -\frac{m\,dr}{r^2}.$$

Thus the potential at P due to the whole solenoid is

$$\int_{r_2}^{r_1} -\frac{m\,dr}{r^2} = m\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

This is what would be due to two poles of strengths m and $-m$ placed at A and B.

If a simple solenoid forms a closed curve it produces no magnetic effect at any point.

DEF.: **A given distribution of magnetism is said to be SOLENOIDAL if the substance in which it exists can be divided into simple solenoids, which either form closed curves, or which have their ends in the surface of the substance.**

For a distribution to be solenoidal, the condition is that there should be no volume-density of magnetism. That is, we have the condition

$$\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} = 0.$$

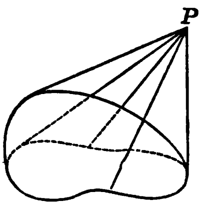
17. DEF.: **A MAGNETIC SHELL is an indefinitely thin sheet magnetized everywhere in the direction normal to itself.**

DEF.: **THE STRENGTH of a shell at any point is the product of the intensity of magnetization and the thickness of the shell at that point.**

A shell is called *Simple* or *Complex* according as its strength is the same throughout or varies from point to point.

Let t be the thickness of a shell at a given point, M the magnetic moment of an element of area dS at this point, I the intensity, and Φ the strength at this point.

Then $$\Phi = It,$$
and $$ItdS = M;$$
$$\therefore \quad M = \Phi dS.$$



Suppose we have a simple shell of strength Φ. Let ω be the solid angle subtended at a point P by the contour of the shell, ω being reckoned positive or negative according as on the whole the positive or negative surface of the shell faces P.

Now consider an element of area dS of the shell, at a distance r from P, whose magnetic axis makes an angle ϵ with r, and which subtends a solid angle $d\omega$ at P. The potential at P due to this element is

$$\frac{\Phi dS}{r^2}\cos\epsilon = \Phi d\omega,$$

for $d\omega$ has the same sign as $\cos\epsilon$.

Thus the potential at P due to the whole shell is

$$\Phi\,\omega.$$

Thus the magnetic action at any point, due to a shell of given strength, merely depends on the contour of the shell.

Suppose a magnetic pole of strength m to be at a point P just outside a shell of strength Φ on its positive side, P being a point at which the contour of the shell subtends a solid angle ω. Let the pole m be moved round the edge of the shell, and brought up to a point Q just opposite to P. At Q a solid angle $4\pi - \omega$ is subtended by the contour of the shell, but Q is faced by the negative side. Thus the potential at Q is $\Phi(\omega - 4\pi)$.

Thus during the motion of the pole work is done on it by the shell equal to $4\pi\Phi m$.

A simple magnetic shell of strength Φ which forms a closed surface produces no potential at any point outside, and a

potential $\pm 4\pi\Phi$ at every point inside, according as the positive side is inside or outside. Thus such a shell will exert no action on a magnetic pole placed either outside or inside it.

18. DEF.: **A given distribution of magnetism is said to be LAMELLAR if the substance in which it exists can be divided into simple magnetic shells, which either form closed surfaces, or have their edges in the surface of the substance.**

If a distribution is lamellar, then in passing from a fixed point to another point, by any path whatever, the algebraic sum of the strengths of the elementary shells we pass across, reckoned positive or negative, according as we pass across any shell from negative to positive side, or *vice versa*, is constant for all paths. Let this be ϕ. Suppose the element of path ds makes an angle θ with the direction of magnetization at any point; and let the intensity of magnetization be I. Then the integral of $I\,ds\cos\theta$ from the first point to the second must be equal to ϕ.

$$\text{Now}\quad I\cos\theta = A\frac{dx}{ds} + B\frac{dy}{ds} + C\frac{dz}{ds}.$$

$$\text{Thus}\quad \int\left(A\frac{dx}{ds} + B\frac{dy}{ds} + C\frac{dz}{ds}\right)ds = \phi,$$

the integral being taken between the first and second points.

Therefore we must have

$$A = \frac{d\phi}{dx},\quad B = \frac{d\phi}{dy},\quad C = \frac{d\phi}{dz}.$$

19. To find the mutual potential of a simple shell and a given magnetic system; that is, the work done on the shell as it moves off from its given position with respect to the system to an infinite distance.

If we have an indefinitely short magnet of strength m, length ds, and moment M, with its negative and positive poles at points in a field at which the potential has the values V and V', then the potential of the magnet in the field is

$$m(V' - V) = m\frac{dV}{ds}ds = M\frac{dV}{ds},$$

where s is measured along the positive direction of the axis of the magnet.

Now let Φ be the strength of the shell, V the potential due to the field at any point on its surface, and n the normal at any point measured in the positive direction of magnetization. Then the potential of an elementary area dS of the shell is

$$\Phi dS \frac{dV}{dn}.$$

Now if N is the normal magnetic intensity at dS due to the field,

$$N = -\frac{dV}{dn}.$$

Thus the potential of the shell in the field is

$$-\Phi \iint N dS.$$

This is the product of the strength and the number of lines of force due to the field passing through the contour of the shell in the negative direction.

This shows that the action of the field on the shell depends only on the form and position of the edge of the shell.

CHAPTER II.

Magnetic Induction.

1. It is frequently useful to specify the magnetic condition at a point inside a magnetic substance by means of a quantity that depends on the magnetization of the substance at that point, and also on the magnetic intensity that exists at this point, and is itself due, in part, to the magnetization. This quantity is called the *magnetic induction* at the point. It will be found to be of great importance in the consideration of the induction of electric currents. Its definition is as follows :—

Def.: **THE MAGNETIC INDUCTION at any point of a magnetized body is the magnetic intensity that would exist at that point, if an indefinitely thin sheet of the substance containing the point and everywhere at right angles to the direction of magnetization were removed.**

Suppose the intensity of magnetization at the point to be I, having the components A, B, C; and the components of the magnetic intensity at the point due to the given distribution to be α, β, γ. Let the components of the Magnetic Induction be a, b, c.

When the indefinitely thin sheet is removed with its infinitesimal quantity of magnetism, it will leave on the two surfaces of the magnetized body on the two sides of the given point surface-distributions given by the formula

$$\sigma = lA + mB + nC.$$

These will in this case be I and $-I$ on the two sides of the given point. And these will produce a magnetic intensity $4\pi I$ at the point, normal to the sheet, or in the direction of magnetization. This must be compounded with the original

intensity at the given point due to the given distribution, the components of which are α, β, γ. It must be noticed that this intensity is due to the distribution actually existing on the body as well as to any other distributions in the field.

We may suppose that before the removal of the sheet the two distributions I and $-I$ were coincident, and thus cancelling each other, but now, after the removal of the sheet, we have them on opposite sides of the point, producing at the point the intensity $4\pi I$, everything else remaining practically the same.

Thus we have for the components of magnetic induction

$$a = \alpha + 4\pi A,$$
$$b = \beta + 4\pi B,$$
$$c = \gamma + 4\pi C.$$

At a point where there is no magnetized substance, the induction is the same as the intensity.

2. Def.: **A line drawn so that its direction at every point coincides with the direction of the magnetic induction is called a LINE OF MAGNETIC INDUCTION.**

Def.: **A tube having its sides all formed of lines of magnetic induction is called a TUBE OF MAGNETIC INDUCTION.**

Where there is no magnetized substance, these lines and tubes coincide with the lines and tubes of force, which are produced by the distribution whose components of intensity are α, β, γ.

3. The surface-integral of normal magnetic induction over any closed surface is zero.

Let S be a closed surface drawn in a magnetic field; l, m, n the direction-cosines of the normal, measured outwards, of an element dS of the surface, a, b, c the components of magnetic induction at dS. Then we have to show that

$$\iint (la + mb + nc)\, dS = 0.$$

That is,

$$\iint (l\alpha + m\beta + n\gamma)\, dS + 4\pi \iint (lA + mB + nC) dS = 0.$$

Now let M be the entire quantity of magnetism contained within the surface S. Then, just as in the case of the corresponding proposition in electrostatics, we have

$$\iint (l\alpha + m\beta + n\gamma)\,dS = 4\pi M.$$

To find the value of the other term, let us suppose all the magnetic substance inside S isolated from that outside by removing an indefinitely thin sheet of the magnetic substance coinciding with S wherever S intersects any magnetic substance. Then we have magnetized bodies inside S insolated from all others, so that the entire quantity of magnetism inside S is zero. That is, the surface-distribution produced by removing the sheets is $-M$.

But the surface-density on the magnetic substance just inside dS which we have produced by removing the sheets is

$$lA + mB + nC.$$

Thus we have

$$\iint (lA + mB + nC)\,dS = -M.$$

Thus the surface-integral

$$\iint (la + mb + nc)\,dS = 4\pi M - 4\pi M = 0.$$

4. It follows that if we have any closed curve, the surface-integral of normal-induction over any surface having this as its contour is constant, and may be called the *quantity of magnetic induction* through the curve. It is represented by the number of lines of magnetic induction passing through the curve.

Also the quantity of magnetic induction across any section of a tube of induction is constant throughout the length of the tube. For the quantity of induction across the sides is zero.

5. INDUCED MAGNETIZATION. It is found that if a piece of iron or steel be placed in a magnetic field it becomes magnetized. Other bodies exhibit the same property too, but to a very much smaller extent than iron or steel. The magnetization which a body possesses under these circum-

stances is said to be *induced*; and the body is said to be magnetized by induction.

When a piece of iron or steel is magnetized by induction and then removed from the field it is found to lose more or less of its magnetization. This portion is called temporary magnetization, and that which is retained is called permanent magnetization. The magnetization induced in soft iron or steel, especially soft iron, can be very great, but very little of it is retained after the body is removed from the magnetizing field. On the other hand, cast-iron and well-tempered steel are not readily magnetized, but when magnetized retain their magnetization for a long time. And as a rule the softer a specimen of steel or iron is the better magnet can it be made while under the influence of the magnetizing field, but the less magnetization does it retain when removed from the field.

The best material to make permanent magnets of is the hardest and best tempered steel.

6. If a homogeneous, isotropic substance is placed in a magnetic field, it becomes magnetized at every point in the direction of the magnetic intensity at that point, and with an intensity of magnetization, at least for weak fields, proportional to the magnetic intensity.

When the positive direction of the induced magnetization is the same as that of the magnetic intensity, the substance is called *Magnetic* or *Paramagnetic*; when it is opposite, the substance is called *Diamagnetic*.

Soft iron is the best example of a magnetic substance, and bismuth of a diamagnetic substance.

Thus the magnetization at any point is found by multiplying the magnetic intensity by a constant κ depending on the substance, which is positive for magnetic and negative for diamagnetic substances.

Thus if H is the magnetic intensity at any point of a homogeneous isotropic substance, the induced magnetization is κH, and the magnetic induction is $(1+4\pi\kappa)H$. This may also be denoted by μH, where $\mu=1+4\pi\kappa$.

DEF.: **THE COEFFICIENT OF INDUCED MAGNETIZATION of a substance is the ratio of its magnetization to the magnetic intensity.**

This is the quantity κ. It is also called the **SUSCEPTIBILITY** of the substance.

DEF.: **THE MAGNETIC INDUCTIVE CAPACITY of a substance is the ratio of the induction to the magnetic intensity.**

This is the quantity μ. It is also called the **PERMEABILITY** of the substance.

7. Suppose a substance *of constant susceptibility* κ to be placed in any magnetic field. If U is the complete magnetic potential at any point, due to the magnetism induced in the body, as well as to the other magnetism present, we have for the components of magnetization,

$$A = -\kappa \frac{dU}{dx}, \quad B = -\kappa \frac{dU}{dy}, \quad C = -\kappa \frac{dU}{dz}.$$

Thus $A\,dx + B\,dy + C\,dz$ is a complete differential. Or *the induced magnetization is lamellar*.

Again, if ρ is the volume density at any point,

$$\begin{aligned}\rho &= -\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right)\\ &= \kappa\left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2}\right)\\ &= -4\pi\rho\kappa.\end{aligned}$$

Now $1 + 4\pi\kappa$ is never zero. Thus $\rho = 0$. Or *the induced magnetization is solenoidal.*

8. If a piece of iron or steel is placed in a magnetic field whose intensity is continually increased the induced magnetization of the substance will not be constantly proportional to the intensity of the field. Thus the susceptibility is not constant. A state of the magnetic substance is gradually reached in which an increase of the intensity of the field hardly increases the magnetization of the substance. This is called the state of *saturation* of the substance.

9. Hysteresis. In addition to this the induced magnetization of a piece of iron or steel will not depend merely on the magnetic intensity of the field in which it is, provided that intensity is small; but will also depend on the manner in which the magnetic substance has been previously treated, that is on its past history. Suppose the intensity is gradually increased, the magnetization increasing in consequence. Let H and I be corresponding values. After the intensity has considerably passed the value H let it be gradually lowered again. The magnetic substance tends to retain its magnetization, and when the intensity is again passing through the value H, the magnetization will be greater than I. Thus for a moderate value of the intensity, the magnetization can have, on account of the retentiveness of the substance for magnetism, any value whatever within certain limits depending on the past history of the substance. If the intensity is made large, either positive or negative, the corresponding magnetization will have a definite value. This phenomenon is called *magnetic hysteresis*.

10. The behaviour of a given substance, magnetic or diamagnetic, when brought into a magnetic field may be explained by reference to the sign of the coefficient κ.

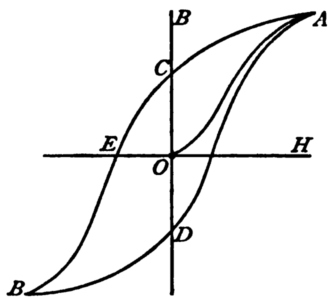
If we bring a bar of unmagnetized iron into a magnetic field it will become magnetized in the positive direction of the lines of force of the field. And since the positive magnetism in the iron is urged along the lines of force of the field in the positive direction, and the negative in the opposite direction, the bar will set itself, if freely suspended, along the lines of force of the field.

On the other hand, if we bring a bar of bismuth into a magnetic field, it will become magnetized in the negative direction of the lines of force of the field. Thus whenever it is inclined to the lines of force of the field at any angle less than a right angle there will be a tendency to drive it still further away from these lines; and, if freely suspended, it will set itself across the lines of force of the field.

11. Magnetic hysteresis may be represented by a curve in

which abscissae denote field strengths, and ordinates denote corresponding values of the magnetization, or of the induction.

Consider the case of an isotropic body, so that everywhere the intensity I, and the induction B, will be in the same direction as the field H. Starting with the substance in the natural state, let the field be gradually increased to a maximum value, then gradually diminished to a lower value, which may be of opposite sign to the first; carried again to the highest value; back to the lowest, and so on.



B will first rise, the curve giving the relation of B to H being OA. After the value of H has been increased and diminished a few times, as stated, the variations of B with H become *cyclical*; that is, the point showing the relation of B to H moves continually on the same closed curve $ACBDA$.

12. Consider the performance of a complete cycle. If we had always the same values of I and of B for a given value of H, the work done by the field, being represented by $\int H dI$ over the cycle, would be zero.

Because of the hysteresis energy is expended on the whole by the field in a cycle; and the amount of this energy per unit volume is given by the value $\int H dI$ taken over the cycle.

Since $B = 4\pi I + H$ this integral becomes

$$\frac{1}{4\pi}\int H dB - \frac{1}{4\pi}\int H dH = \frac{1}{4\pi}\int H dB, \text{ in a closed curve.}$$

Heat is developed by the changes of magnetization. Suppose this to be given off by the substance as it is produced, or the substance kept at a constant temperature. Then the substance is in just the same condition at the beginning and end of a cycle.

The energy expended by the field all goes to produce heat; and the heat generated in a cycle is $\frac{1}{4\pi}\int HdB$ over the cycle. Thus it is represented by $1/4\pi$ of the area of the cycle.

In the same way, if we draw the curve to co-ordinate I and H, the energy expended by the field in a cycle, being $\int H\,d\,I$, will be represented by the area of the cycle.

When the substance has gone through a complete series of changes, represented by a closed curve on the diagram, the work done by the field will always be represented by the area of the curve on the I and H diagram, or by $1/4\pi$ of the area on the B and H diagram. But only when the changes have become cyclical, in the sense explained above, can we say that the heat developed is represented by the same quantity. For in any other case the substance will not be in quite the same condition at the beginning and end of a series of changes; and so, although it may have the same values of I and H, and the same temperature, we cannot say that it has the same intrinsic energy.

13. The curve drawn to co-ordinate I and H will be similar in appearance to that for B and H. Taking the above figure to represent such a curve, suppose it cuts the positive direction of the I axis in C, and the negative direction of the H axis in E.

OC represents the magnetization that remains when the magnetizing field has been removed, and is called the *residual magnetism.*

OE represents the magnitude of the reversed intensity that must be applied to remove the magnetization, and has been called by Dr. Hopkinson the *coercive force.* This expression, which has been very loosely applied, has thus received an exact meaning.

CHAPTER III.

EARTH'S MAGNETISM.

1. WE have already seen that a magnet at any place on the earth's surface tends to set itself with its magnetic axis in a certain definite direction. Thus the magnet is in a magnetic field. And one would find a magnetic field all over the surface of the earth with its intensity and direction varying continuously from point to point.

2. A magnet suspended so that its position of equilibrium may be horizontal will in most latitudes rest with its magnetic axis pointing about north and south. The vertical plane which contains the magnetic axis of such a magnet when at rest is called the **MAGNETIC MERIDIAN** at the given locality.

The field due to the earth in a given locality is completely known if we know what are called the *Earth's Magnetic Elements* for the given locality. These are:

(1) The **DECLINATION**: that is the angle between the magnetic and geographic meridians, or the angle between the magnetic axis of a freely suspended needle and the line drawn due north and south, the magnetic axis being supposed to be in a horizontal plane.

(2) The **INCLINATION** or **DIP**: that is the angle which the lines of force due to the field make with the horizontal plane, or the angle which the magnetic axis of a needle turning on a horizontal axis set at right angles to the magnetic meridian, and passing through its centre of gravity, so that its magnetic axis may move in the magnetic meridian, makes with the horizontal plane.

(3) The INTENSITY: that is the force in dynes with which a unit magnetic pole in the given locality would be acted upon.

3. These elements not only vary from point to point of the earth's surface; but in the same place are subject to small periodical variations called the *Magnetic Variations.*

It is usual to find, at a given place, the declination, the inclination, and the horizontal component of the intensity. From these the total intensity can be determined.

The horizontal component of the intensity of the field due to the earth, in any given locality, is denoted by the letter H.

4. In any given place the value of H may be determined by a method due to Gauss. The following is a sketch of this method.

Suppose we have a bar magnet, whose moment of inertia A about some straight line through its centre of gravity and perpendicular to its magnetic axis is known. Let M be the magnetic moment of the magnet, supposed to be unknown. We have to perform two sets of experiments, which we may call the *vibration* experiments and the *deflexion* experiments, for finding, respectively, the values of MH and M/H.

(1) *The vibration experiments.*

Let us suspend the magnet by a torsionless fibre so that its axis is horizontal, and set it to make small oscillations, under the action of the earth's magnetic field, about the axis about which the moment of inertia is A, this axis being vertical.

When the magnet is deflected by an angle θ from its position of equilibrium the moment of the force tending to restore equilibrium is $MH \sin\theta$. So that for oscillations of very small amplitude the equation of motion of the magnet is

$$A\frac{d^2\theta}{dt^2} + MH\theta = 0.$$

Let us observe the time T of a complete vibration. Then we have

$$T = 2\pi\sqrt{\frac{A}{MH}};$$

$$\therefore \quad MH = \frac{4\pi^2 A}{T^2}.$$

(2) *The deflexion experiments.*

Let us take a small magnetic needle, freely suspended, so as to swing horizontally and to be at rest with its axis in the magnetic meridian. Set the magnet so that its axis is perpendicular to the magnetic meridian, and when produced in the positive direction it passes through the centre of the needle; the centre of the magnet being at a distance r from that of the needle, r being great compared with the dimensions of the magnet. This will deflect the needle from its position of equilibrium. Let the observed deflexion be a.

Now the potential due to the magnet at the needle is M/r^2. Thus the intensity of the field due to the magnet, at the needle, which field is at right angles to the magnetic meridian, is $2M/r^3$. Therefore the deflecting couple acting on the needle, m being the magnetic moment of the needle, is

$$\frac{2M}{r^3} m \cos a.$$

And the couple with which the earth's field acts on the needle, to turn its axis back to the meridian, is

$$Hm \sin a.$$

These two couples are equal. And thus we get

$$\frac{M}{H} = \frac{r^3}{2} \tan a.$$

From these two results we can calculate the value of H. M too can be determined if required.

To obtain the deflexion a it is best to place the magnet on both sides of the needle, with its axis pointing towards the needle in the same way in both cases, and to take the mean value of the deflexions thus observed for a.

5. To get as accurate a value for H as possible by means of this method it is necessary to make a large number of corrections.

One important correction is that which has to be made on account of the dimensions of the magnet and the needle not being so small that we can neglect them in comparison with r.

Let us suppose first that we have an ideal magnet and an ideal needle, the distances between the poles being respectively $2a$ and $2b$. Then, retaining negative powers of r only as far as r^{-2}, we can show that

$$\frac{M}{H}\left\{1 + \frac{1}{r^2}(2a^2 - 3b^2 + 15b^2 \sin^2 a)\right\} = \frac{r^3 \tan a}{2}.$$

Thus, very approximately, P being a constant,

$$\frac{M}{H} = \left(1 - \frac{P}{r^2}\right)\frac{r^3 \tan a}{2}.$$

Now, with an ordinary magnet and needle, the magnet being magnetized as uniformly as possible, the same approximate relation will hold. r is measured from the centre of figure of the magnet; and, to correct for small inequalities in the distribution of magnetism with respect to this point, the magnet's direction may be reversed and the mean of the deflexions taken. The same observations may be made with the magnet on the other side of the needle to correct for uncertainty with regard to the position of the centre of the needle.

The constant P is estimated by using two different values of r.

In this way a fairly accurate value can be found for H without making any further corrections.

If the magnet used is a uniform cylinder, of mass μ grammes, length l cms., and diameter d cms., its moment of inertia is

$$A = \mu\left(\frac{l^2}{12} + \frac{d^2}{16}\right).$$

6. Astatic Combinations. A combination of magnets rigidly attached to each other, and such that when freely suspended in a uniform magnetic field it does not tend to set in any definite direction, is called an astatic combination.

To make an astatic combination of two magnets, each being set on at right angles to a common axis, it is obvious that the moments of the magnets should be equal, and their axes parallel and turned in opposite directions.

Suppose we have a nearly astatic combination, the moments

of the magnets being nearly equal, being M and $M+\delta M$, and their axes making a small angle a with each other. To determine how the combinations will set.

When suspended under the action of the earth's field so that the axis of the magnet M makes an angle θ with the magnetic meridian, the moment of the forces tending to set the combination in the meridian is

$$\begin{aligned} &H[M\sin\theta-(M+\delta M)\sin(\theta+a)] \\ =\ &H[M\sin\theta-(M+\delta M)(\sin\theta+a\,.\cos\theta)] \\ =\ &-H[aM\cos\theta+\delta M\sin\theta], \end{aligned}$$

retaining only small quantities of the first order.

Thus when the compound magnet is in equilibrium,

$$\tan\theta=-\frac{aM}{\delta M}.$$

We may notice two particular cases.

(1) If the moments of the magnets are exactly equal, but their axes are not parallel, $\delta M=0$, and the combination will set itself across the meridian.

(2) If the axes are exactly parallel, but the moments are not equal, $a=0$, and the combination will set itself in the meridian.

PART III.

CHAPTER I.

THE ELECTRIC CURRENT.

1. IF we have two points A and B at different potentials, A being at the higher potential, and connect them by a conducting wire, electricity will flow from A to B. If we suppose A and B to be points on two insulated conductors charged to different potentials, when they are connected by a wire, the electricity on the conductors will soon redistribute itself bringing them to the same potential, and the duration of the flow along the wire will be very short indeed. But if by any means we can keep A at a higher potential than B, then there will be a permanent flow or *current* of electricity from A to B. And the quantity of electricity that passes from A to B per second will depend on the difference of potentials between A and B, and the nature of the conductor with which we connect A and B.

One of the most convenient ways of obtaining a steady current of electricity is by means of a Voltaic Battery. This is an arrangement in which the energy produced in chemical combinations is used to furnish the work necessary for driving the current.

2. One of the best-known forms of battery, and one of the best suited for producing a steady current, is Daniell's. This

may be made in the following way. A porous vessel containing a saturated solution of copper sulphate is put into another vessel containing a saturated solution of zinc sulphate. Into these two solutions are dipped plates of copper and zinc, respectively, having copper wires soldered to their upper ends which project out of the liquids. If these wires be joined across by a conductor, a flow of electricity will take place through it from the copper plate to the zinc plate of the battery. At the same time the chemical energy of the battery will be used up. The zinc will be dissolved in the sulphate of zinc; the quantity of sulphate of zinc will increase, and the quantity of sulphate of copper will diminish; and copper will be deposited on the copper plate. We shall see later on that the amount of each of these actions is exactly proportional to the quantity of electricity that has passed.

This arrangement is called a Daniell's cell. A battery may be made of several cells by joining the zinc of the first to the copper of the second and so on. Then the wire joined to the copper of the first cell is called the positive pole of the battery, and that joined to the zinc of the last cell is called the negative pole. On joining the poles by a conductor a current always flows from the positive to the negative pole outside the battery.

3. DEF.: **THE ELECTROMOTIVE FORCE of a battery is the excess of electrical potential of its positive pole over that of its negative pole when no current is flowing through the battery.**

This quantity is usually denoted by the symbol E. M. F.. The E. M. F. of a Daniell's cell is much more constant than that of most cells. In most other cells the E. M. F. falls off considerably when they are used.

The difference of potential, sometimes called the P. D., between two points is also called the electromotive force, or E. M. F. between them.

4. The measure of an electric current or the *strength* of the current is the number of units of electricity passing per second in the current. But it is found convenient to adopt a different unit of electricity in the case of current electricity from that

which is adopted in electrostatics. We shall now consider a convenient unit of electric current.

We must first refer to the chemical actions of the current. If an electric current is passed through a chemical compound it produces decomposition. This action is called Electrolysis. If, for instance, a current is passed through water, with which a little sulphuric acid has been mixed to make it conduct well, the current being led into the water by a platinum plate and out by another (these plates being called the electrodes), the water will be decomposed into hydrogen and oxygen. We shall later on speak more fully of the laws of Electrolysis and the evidence on which they rest; but one important result proved by Faraday is that the amount of any chemical decomposition produced by Electrolysis is exactly proportional to the quantity of electricity that has passed, and is independent of the quantity per second that has passed, or any other circumstances.

The proof of this did not depend on measures of the currents made by any other process. Faraday passed the same current through several electrolytic cells all in a series, so as to be sure that the same quantity of electricity had passed through each: the electrodes were made of various sizes, so as to vary the current density. The same amount of electrolytic decomposition was found to have taken place in each cell. In another experiment the current after passing through a cell A was made to divide itself between two cells B and C. The amount of decomposition in B and C together was found to be exactly equal to that in A.

Thus if we have a steady current flowing and we can make it pass through a chemical compound and decompose it, and we can measure the amount of decomposition produced per second, we may take this as a measure of the current. Suppose, for instance, we decompose water and measure the quantity of hydrogen liberated per second. This is proportional to the strength of the current.

5. But it is by means of another effect produced by the electric current that the measure of its strength is determined. Örsted

discovered in 1820 that an electric current flowing in a straight wire when brought near to a magnetic needle tends to set the needle at right angles to the current. In fact any electric circuit carrying a current produces a certain magnetic field. And it has been proved by experiment that the intensity of this field is at every point exactly proportional to the strength of the current as measured by Electrolysis. The strength of a current is then measured by the magnetic intensity which a certain conductor carrying that current produces at a certain point.

6. The way in which a straight conductor carrying a current tends to deflect a magnetic needle is given by the following rule of Ampère's :—

If we suppose a man to swim with the current in the conductor, and to turn so as to look at the magnetic needle, the conductor tends to deflect the N. pole of the needle to his left hand.

An instrument for detecting electric currents may be made by winding several turns of wire insulated from each other in a coil, and suspending a magnetic needle inside the coil. Then if a current is sent through the wire of the coil it follows from Ampère's rule that each portion of the wire tends to deflect the needle in the same way. By multiplying the number of turns in the coil the instrument may be made more sensitive.

An instrument for detecting or measuring electric currents is called a Galvanometer.

7. Before we can define the unit of current we must consider more closely the magnetic actions of electric circuits.

The precise action between a magnetic system and any portion of an electric circuit carrying a current may be deduced from experiments of Ampère's and Weber's, which have shown that the magnetic action of a small plane circuit at distances from it which are great compared with its dimensions, is the same as that of a small magnet whose axis is normal to the plane of the circuit, and whose moment is proportional to the product of the area of the circuit and the current which it carries. As the unit of current has not yet been defined, we

may take the moment of the equivalent magnet to be *equal* to the product of the area of the circuit and the current which it carries. We shall see later on what precise definition of unit current this leads us to.

The action of an electric current depends, in a way that we shall consider, on the medium; unless the contrary is specified, we shall suppose the medium to be air under standard conditions.

The direction of the current in the circuit and the positive direction of the axis of the equivalent magnet are connected by the following important rule:—

A right-handed screw placed along the axis of the magnet and turned in the direction of the current will move in the positive direction of the axis of the magnet.

This we may call the *right-handed screw law.*

8. We may now prove that the magnetic action of any circuit carrying a current is the same as that of a simple magnetic shell whose contour coincides with the circuit, whose strength is equal to the current in the circuit, and whose direction of magnetization is connected with the direction of the current according to the right-handed screw law.

Let us consider the magnetic action of the circuit at a point P. Draw any surface S having the circuit for its contour, and not passing through P. Draw two series of lines on S to cut it up into elementary surfaces each one of which is small compared with its distance from P, and the radii of curvature of S at it. Let i be the current in the given circuit. We may suppose a current i to pass round each one of the elementary circuits into which S is divided, for the two currents passing along the common edge of any two adjacent elements are in opposite directions and neutralize each other, so that the elementary circuits each carrying the current i are equivalent to the given circuit carrying the current i.

Now suppose a magnetic shell of uniform strength i coinciding with S. The magnetic moment of an elementary portion dS is idS. Thus, by the experiments referred to, this produces

the same magnetic effect at P as the elementary circuit dS carrying the current i. So that the entire shell is equivalent with regard to its magnetic action to all the elementary circuits, and therefore to the given circuit carrying the current i.

9. We see that a given closed electric circuit produces the same magnetic action at all points as any simple magnetic shell having its contour coinciding with the circuit, and of strength equal to the current in the circuit. This may be called an *equivalent magnetic shell* of the given circuit.

We shall call the positive direction of magnetization of the equivalent shell of a given circuit, as given by the right-handed screw law, the positive direction through the given circuit.

10. We can now find the potential of a circuit in a magnetic field.

First suppose we can replace the circuit by an equivalent magnetic shell. Then the required potential is the same as that of the shell; that is, it is the product of the strength of the current and the quantity of induction passing through it in the negative direction.

Next suppose that no surface can be drawn having the circuit for contour without cutting across some substance in the field. Draw such a surface cutting the substances it meets with everywhere at right angles to the lines of induction; and remove from these substances indefinitely thin sheets coinciding with the surface drawn. When this is done the intensity at any point of the surface we have drawn is equal to the induction of the given field at that point. We can now place a magnetic shell in the field coinciding with this surface and equivalent to the given circuit: and the potential required is the potential of this shell. Thus the potential of the circuit is the product of the strength of the current and the quantity of induction passing through it in the negative direction.

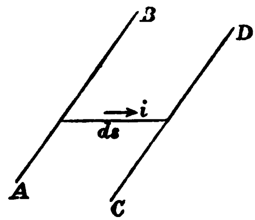
This may also be measured by the product of the strength of the current and the number of lines of induction passing through the circuit in the negative direction.

Thus the work done on the circuit by the field during any

change of position, the current remaining constant, is the product of the strength of the current and the *increase* in the magnetic induction through the circuit in the positive direction.

11. We can now find the mechanical action, or *Electromagnetic Force,* acting on an element of a circuit, carrying a current i, in a given magnetic field.

Consider a small element ds of a circuit carrying a current i. Let the magnetic induction at ds be H: and let its direction make an angle θ with ds. Now suppose ds capable of sliding parallel to itself so that its ends move on two parallel metallic rods AB, CD. As ds slides in this way the field does work on it measured by the product of i and the quantity of induction taken in by ds



in the positive direction. This is a positive quantity, as appears from the right-handed screw law, when an observer turning so as to look successively in the directions of the current, the induction, and the motion of ds, looks round in the clock-wise direction. For a given amount of displacement of ds the maximum quantity of work will be done if ds moves so as to include the maximum quantity of magnetic induction. This will be when the motion of ds is at right angles to itself and to the lines of force. Therefore this is the direction of the Electromagnetic force acting on ds. Now if ds moves in this direction through a very small distance n, the work done by the electromagnetic force F is $i . H . n ds \sin \theta$.

Thus $$Fn = i . H . n ds \sin \theta.$$
$$\therefore \quad F = iH ds \sin \theta.$$

And the direction of F is at right angles to ds and to the magnetic induction at ds.

12. Suppose we have a single magnetic pole m acting on ds. Let r be the distance of m from ds, and θ the angle between

r and ds. The induction at ds is m/r^2. So that the force on ds is

$$F = \frac{mi\,ds \sin\theta}{r^2}.$$

And therefore the force with which ds acts on m is a force equal to this and in the opposite direction or $-F$. That is, the magnetic intensity due to ds at the pole m, or the force on unit pole, is

$$\frac{i\,ds \sin\theta}{r^2}.$$

The forces F and $-F$ with which m acts on ds, and ds acts on m, respectively, are not directed along the same straight line. Let us consider in what sense we may say that $-F$ is the force with which ds acts on m.

We may suppose m to be the only magnetism in the field, by supposing it to be a pole of an infinitely long uniformly magnetized magnet, the other pole $-m$ being at an infinite distance away. Then we have shown that F is the force with which m acts on ds. But it must be remembered that ds alone does not act on m, every portion of the closed circuit acts on m. If we consider the action of m on the entire closed circuit, and the reaction of the circuit on m, these would be two equal and opposite forces along the same straight line. Now the action on the circuit would be obtained by compounding all such forces as F. Therefore the reaction on m would be obtained by compounding all such forces as $-F$. In this sense we may say that the reaction of ds on m is $-F$; for if we compound all such forces as $-F$, supposed to be due to each element of the circuit, we get the true reaction on m.

In this sense we may say that the magnetic intensity at m due to ds is

$$\frac{i\,ds \sin\theta}{r^2}.$$

For by compounding all such intensities due to each element of the circuit we get the true intensity at m.

13. That the action and reaction of m and the circuit are

equal and opposite and along the same straight line follows from the fact that two indefinitely small quantities of magnetism q and q' act on each other with forces, both equal to qq'/r^2, in opposite directions, and along the straight line joining them. Now the circuit is equivalent to a certain magnetic system: thus when we consider the circuit and the pole m together the forces arising from their mutual actions just cancel each other.

The same thing may be proved experimentally by showing that a system composed of a circuit and a magnet rigidly connected does not tend to displace itself; that is, there is no resulting force or couple arising from the mutual actions of its parts.

14. Suppose we have a wire of length l bent into a circular form of radius r and carrying a current i. Let there be a pole of strength m at the centre of the circle. We may suppose each element of length ds of the circular circuit to act on the pole m with a force urging it along the axis of the circle in the direction given by the right-handed screw law and equal to

$$\frac{m\,i\,ds}{r^2}.$$

Thus the entire force on m is, in dynes,

$$\frac{m\,i\,l}{r^2}.$$

From this we see that if the current is of unit strength, the pole of unit strength, the radius of the circle one centimetre, and the length of arc one centimetre, the force with which the pole will be acted on will be one dyne. Thus we can give the following definition of unit of current.

DEF.: **THE UNIT OF CURRENT is the current which, flowing in a conductor in the form of a circular arc one centimetre long, and having a radius of a centimetre, acts with a force of one dyne on a unit magnetic pole placed at the centre of the circle.**

It should be noticed that the direction of the force with which

a magnet pole is acted on, due to an electric circuit, as given by the right-handed screw law, coincides with that given by Ampère's rule.

15. Let us consider the work done by a closed circuit carrying a current i on a magnetic pole m, which traverses a closed path so drawn as to be linked with the circuit, that is, to pass through it once and round outside. And suppose the motion of the pole is in the positive direction of the magnetic field of the circuit.

Replace the circuit by an equivalent shell, and let the pole start from a point P just outside the + face of the shell and travel up to a point Q just outside the −face, and opposite to P, so that P and Q are indefinitely close to each other; and in the limit the work done in the path from P to Q is equal to that done *by the circuit* in the same path closed up by joining P and Q. Now suppose the circuit subtends a solid angle ω at P. Then the −face of the shell subtends a solid angle $4\pi-\omega$ at Q. Thus the potentials of the shell at P and Q are $i\omega$, and $-i(4\pi-\omega)$.

Therefore the work done by the shell on unit pole in the path from P to Q is

$$i\omega+i(4\pi-\omega)=4\pi i.$$

The work done by the circuit on pole m in the closed path is $4\pi m i$.

This result is of great importance. It will be useful to extend it as follows.

Suppose the closed path of the pole to wrap round the circuit n times. This may be replaced by n separate closed paths, each linking the circuit once, and such that the additional portions which have been introduced to make up the n paths always cut each other out in pairs, being coincident, and being traced in opposite directions. Thus the work in the single path is equal to the work in n paths, each linking the circuit once, and is, therefore, $4\pi n m i$.

If the circuit has n' turns the work it does on pole m traversing a path which wraps it round n times is easily seen to be $4\pi n n' m i$. For if the n' turns are close to each other the circuit is mag-

netically equivalent to one of a single turn with current $n'i$. And if the turns are spread out they may be replaced by n' simple circuits, the additional pieces which have been introduced to make up the n' circuits cutting each other out in pairs, since equal currents flow along them in opposite directions.

The work done by a circuit on a pole traversing any closed path which does not link with the circuit may be seen by similar reasoning to be zero.

16. We must now notice an important difference between a circuit and an equivalent shell. The circuit is completely represented in its magnetic effect by the shell at all points, *except where the material of the shell is.* The magnetic intensity due to the circuit is continuous throughout any such path as we have been considering, but that due to the shell becomes discontinuous at the faces of the shell. And at any point in the material of the shell the intensity has a different value, in fact, a value of opposite sign to that which the intensity at the same point due to the circuit has.

The work done by the simple circuit on pole m, in a closed path once linking it, is $4\pi mi$. But the work done on the pole in the same path by an equivalent shell, if we suppose the pole to move through the shell without disturbing its magnetism (as we must suppose to be the case if the shell remains equivalent to the circuit), is clearly zero; being the work done on the moving pole by given attracting and repelling masses as it moves off from a given position and back again to it.

The shell has at a point at which its + face subtends a solid angle ω, a potential $i\omega$. But we cannot say this of the circuit. For this may be made to do any quantity of work on a + unit pole starting from the point, the pole going through the circuit as often as we please; the battery which supplies the current furnishing the energy required. We must consider the potential of the circuit at the given point to have an infinite number of values, the difference between any consecutive two being $4\pi i$.

17. Now suppose there are in the field of the circuit any

magnets and magnetic substances; and let us consider the work done on pole m in a path once linking the circuit. The magnetic forces of the field are now made up of two sets; those that the circuit would alone produce if in a homogeneous field of unit permeability, and those due to the distributions of magnetism in the field partly permanent and partly induced. The work done in the path by the circuit is $4\pi m i$. That done by the magnetic distributions is zero.

Thus the entire work done, in this case as well, on the pole m in the given path is $4\pi m i$.

We may express this as follows. Let H be the resolved part of the entire magnetic intensity along the path s, closed, and once linking the circuit. Then

$$\int H ds = 4\pi i,$$

the integral being taken once round the path and in the positive sense with reference to the circuit.

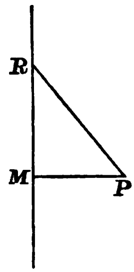
18. An electric circuit will produce just the same field in any homogeneous isotropic indefinitely extended medium. For the intensity at any point will be made up of that which would be produced at the same point by the circuit directly in a medium of permeability unity, and of that due to the distribution induced by the circuit; and, in the case of the medium specified, there is no induced distribution.

On the other hand any arrangement of magnets, say, a shell equivalent to the circuit, will produce different fields in media of different permeabilities, on account of the magnetic distribution such an arrangement will induce in the medium at the surfaces of separation of it and the magnets.

It follows that if a circuit and shell are equivalent in one medium they will not be so in a medium of different permeability.

19. Suppose we have an indefinitely extended linear conductor carrying a current i. To find the magnetic intensity it produces at a point distant r from it.

From the given point P draw PM perpendicular to the conductor. Take a point R on the conductor, such that $MR=x$. Then the intensity at P due to an element dx at R is at right angles to the plane PRM, and



$$=\frac{i dx \sin PRM}{r^2+x^2},$$

$$=\frac{i r dx}{(r^2+x^2)^{\frac{3}{2}}}.$$

Thus the intensity at P due to the entire current is

$$\int_{-\infty}^{\infty} \frac{i r dx}{(r^2+x^2)^{\frac{3}{2}}}.$$

Or, putting $x=r \tan \theta$ we get

$$\frac{i}{r} \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta = \frac{2i}{r}.$$

Again, the intensity at P may be found as follows. We may suppose the circuit of the indefinitely extended straight conductor to be completed by a conductor all of which is infinitely distant from P. For the action of an element of this conductor on a pole at P is inversely proportional to the square of its distance from P, while the length of the conductor is proportional to its distance from P. Thus the action of this infinite conductor on a pole at P may be neglected. And the action of the entire infinite circuit formed is the same at P as that of the straight conductor alone. Now let us suppose that the whole of the completed circuit is in a plane with P. Then the motion of a magnet pole at P either parallel or at right angles to the given conductor will produce no variation of the angle subtended at it by the contour of the circuit, and therefore no work will be done on it. Thus the intensity at P is at right angles to the plane containing P and the given conductor. Let the intensity be I. Then if a pole of strength m moves in a circle round the given conductor keeping at a distance r from it, so that the

intensity at it is always I, and in the direction of its motion, the work done on it is $2\pi r I$, and also $4\pi i$.

$$\therefore I = \frac{2i}{r}.$$

In the same way we may see that if a current of strength i flows in a cylindrical conductor, and is uniformly distributed in it, or distributed so that the density is proportional to the distance from the axis of the cylinder, than the intensity it produces at any point outside the cylinder at a distance r from its axis is $2i/r$.

20. The subject of the equivalence of a circuit and a shell may be approached from a different point of view, as has been done by Biot and Savart, who made experiments to determine the law of magnetic intensity produced by an infinite straight conductor carrying a current. This was done by oscillating a short magnet first under the action of the earth alone, and then under the action of earth and current. In this way they found the law which we have deduced in Art. 19 from Ampère's experiments, that the intensity is proportional to i/r.

Thus the two sets of experiments agree with each other. But it is not difficult, starting from Biot and Savart's experiments, to show that any plane circuit bounded by straight lines, and thus any circuit (by considering it made of indefinitely small plane circuits bounded by straight lines), produces a field as if it had a potential at any point proportional to the solid angle it subtends at that point, and thus acts like a magnetic shell having the same contour.

21. A Tangent Galvanometer is an instrument for measuring directly the strength of an electric current. It consists of a number of turns of wire wound in a circle, with a short magnet suspended at the centre; the plane of the coil being set in the magnetic meridian.

Suppose we have n turns of wire wound in a circle of radius a, the current being brought up to the coil and taken away from it by two wires wound closely together, so that the effects of the

currents in these two on the needle neutralize each other. Let this coil be set in the plane of the magnetic meridian; and let a short needle of moment M be suspended or pivoted at the centre of the coil. Let the horizontal component of the earth's magnetic intensity be H.

Suppose that on passing a current i through the instrument the needle is deflected from its position of equilibrium, and therefore from the plane of the coil, by an angle θ.

Now the length of wire in which the current is flowing is $2\pi na$. Thus the intensity at the centre due to the current is

$$\frac{2\pi nai}{a^2} = \frac{2\pi ni}{a}.$$

Therefore the moment of the couple with which the field due to the current i acts on the needle is

$$\frac{2\pi niM\cos\theta}{a}.$$

This must be just equal to the moment of the couple with which the earth's field acts on the needle. And this is

$$HM\sin\theta.$$

Thus
$$i = \frac{a}{2\pi n}.H.\tan\theta.$$

22. In general, a galvanometer is made of a number of turns of wire forming a coil, and having a space left inside the coil unoccupied by wire, in which a short magnetic needle is suspended. The passage of a current through the coil is shown by producing a magnetic field which causes the needle to be deflected.

A great many galvanometers are used, not for finding the absolute measure of a current, but for detecting the presence of a current; or sometimes for comparing the relative strengths of two currents, so that we do not require to know the dimensions and number of turns in the coil.

Suppose a galvanometer has a very short needle, which is so situated with respect to the coils when in its position of equili-

brium, that we may suppose the magnetic field due to a current in the coils is, in the neighbourhood of the needle, uniform and at right angles to the needle. This field is also proportional to the current i; so that we may denote it by Gi. G is called the *galvanometer constant.*

If this current deflects the needle through an angle θ against the action of the earth's field H, we have, if M is the magnetic moment of the needle,

$$G\,i\,M\cos\theta = H\,M\sin\theta,$$

$$\therefore \quad i = \frac{H}{G}\tan\theta.$$

The constant for a tangent galvanometer is $2\pi n/a$.

The multiplier H/G is called the *reduction factor* for the galvanometer, and is, of course, not constant, depending on the value of the horizontal component of the earth's magnetic intensity.

23. Suppose at any point of the field the current density, or the quantity of current per unit area at right angles to the direction of its flow, is q. Let u, v, w be the components of q.

Let α, β, γ be the components of the magnetic intensity at the same point.

Consider a small rectangle with sides, dy, dz, parallel to the axes of y and z. The quantity of current flowing through this rectangle is $udydz$. And the line integral of magnetic intensity round it is

$$\beta dy + \left(\gamma + \frac{d\gamma}{dy}dy\right)dz - \left(\beta + \frac{d\beta}{dz}dz\right)dy - \gamma dz,$$

or
$$\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)dy\,dz.$$

Thus we have

$$4\pi u\,dy\,dz = \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)dy\,dz,$$

or
$$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz}.$$

Similarly
$$4\pi v = \frac{d\alpha}{dz} - \frac{d\gamma}{dx}$$

and
$$4\pi w = \frac{d\beta}{dx} - \frac{d\alpha}{dy}.$$

CHAPTER II.

STEADY FLOW IN CONDUCTORS.

1. Ohm's Law. If we take a conducting wire AB and maintain its two ends at different potentials so as to get a steady electric current flowing along the conductor, the magnitude of this current will depend on the potentials of the ends of the conductor, and on its size, shape and material. The exact law on which the current depends was investigated experimentally by Dr. Ohm, and may be given as follows.

If a steady current pass through a given conductor kept at a constant temperature the potential-difference of the terminals of the conductor is proportional to the current passing.

Thus, if E denotes the E. M. F. between the terminals of a conductor kept at a fixed temperature, and C the steady current which passes when the E. M. F. E is maintained, we have the relation

$E = C \times$ a constant quantity for the given conductor at the given temperature.

This constant quantity is called the electrical resistance (or resistance simply) of the conductor at the given temperature.

We may define the resistance of a conductor at a given temperature as follows.

DEF.: **THE RESISTANCE of a conductor is the constant quotient obtained by dividing the E.M.F. between its terminals by the value of the steady current flowing through the conductor with that E.M.F.**

In future when the resistance of a conductor is spoken of

it will be supposed that the conductor is kept at some fixed temperature unless the contrary is specified. And it will be supposed that the current passing in a conductor is steady unless the contrary is specified.

2. In considering conductors which have very small cross-section as compared with their length, such as conducting wires, the two ends of the conductor, which may be regarded as points, are supposed to be kept at a certain difference of potential. And these two points at which the current enters and leaves are called the *electrodes* of the conductor. But in some cases the electrodes of a conductor have to be considered as surfaces of finite extent, as when we speak of the resistance of a cylinder of section which is not small as compared with its length. Then the two ends of the cylinder must be taken as the electrodes, and we must suppose, in considering what its resistance is, that the whole surface of one end is kept at the same potential, and the whole surface of the other end at some other potential, the same throughout.

The electrical resistance of a given conductor will of course depend upon what points or surfaces we agree to consider as its electrodes. But in the case of a wire or cylinder, these are always taken to be the ends as explained.

3. Combined resistance of a number of conductors.

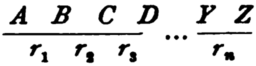
If we have a number of conductors joined together in any way, and their combined resistance is spoken of, it is supposed that two electrodes A and B in the combination are fixed upon by which the current is to enter and leave, and the combined resistance means the resistance of a single conductor having its electrodes at A and B, which would allow the same current to pass for the same E. M. F. from A to B; that is, it is the quantity by which we must multiply the entire current passing through the combination from A to B, to obtain the E. M. F. from A to B.

We shall consider two cases of combinations of conductors.

(1) Suppose a number of conductors joined together 'in series,' that is, end to end in a continuous row. Let $r_1, r_2, r_3, \ldots r_n$ be

the resistances of the conductors, and R the resistance of the combination.

Let AB, BC, CD,...YZ denote the conductors. We require to find the resistance R from A to Z.



The same current must pass through each conductor. Denote it by C. And let V_A, V_B,...V_Z denote the potentials at A, B,...Z.

Then
$$V_A - V_B = Cr_1,$$
$$V_B - V_C = Cr_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$V_Y - V_Z = Cr_n;$$

Thus $$V_A - V_Z = C(r_1 + r_2 + \ldots + r_n).$$

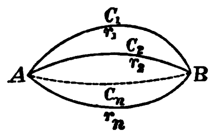
But $$V_A - V_Z = CR.$$

Therefore $$R = r_1 + r_2 + \ldots + r_n = \Sigma r.$$

(2) Suppose a number of conductors joined together 'in parallel,' that is, with all their ends connected to the same two points A and B.

Let r_1, r_2, r_3,.. r_n be the resistances of the conductors, and R the resistance of the combination.

Let C_1, C_2, C_3,...C_n be the currents flowing through the conductors, and C the entire current through the combination.



Let V_A, V_B denote the potentials at A and B.

Then $V_A - V_B = C_1 r_1 = C_2 r_2 = C_3 r_3 = \ldots = C_n r_n$.

Therefore
$$(V_A - V_B)\left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}\right) = C_1 + C_2 + \ldots + C_n = C.$$

$$\therefore \quad V_A - V_B = C . \frac{1}{\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}}.$$

But $$V_A - V_B = CR.$$

Therefore
$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n} = \Sigma \frac{1}{r}.$$

4. As particular cases we have the following.

If there are n conductors each with resistance r arranged in series, their combined resistance is rn.

If they are arranged in parallel their combined resistance is r/n.

It follows from this that if we take a conductor of given material and of uniform cross-section, and vary its length in any manner, its resistance is directly proportional to its length. And if we vary its cross-section in any manner its resistance is inversely proportional to its cross-section.

Thus if we have a conductor of a given material, of uniform cross-section s, and of length l, its resistance expressed in terms of any given unit of resistance is given by the formula kl/s, where k is a quantity depending only on the material and physical state of the conductor, and the unit of resistance used. k is called the *specific resistance* of the material, and may be exactly defined as follows.

DEF.: **THE SPECIFIC RESISTANCE of a given material is the resistance of a portion of it of unit cross-section and unit length.**

5. We have seen that if we have a wire AB, a current will flow along it or not according as its ends A and B are kept at different or the same potentials. Now let the wire be cut across and a battery of E. M. F. E joined up to its parts, so that its positive pole is joined to the part of the wire towards B, and its negative pole to the part of the wire towards A. Then if the parts of the wire are insulated so that no current flows through either part of it or the battery, A and B are at the potentials of the poles of the battery, that is, B is at potential E above that of A. And this is the sufficient condition for no current to pass in the wire which now includes a battery of E. M. F. E.

Let us denote the potentials of A and B by the letters V_A, V_B. Then we have seen that the condition for no current is

$$V_B - V_A = E.$$

Now if the value of $V_B - V_A$ be made greater or less than E,

a current will flow through the wire and battery from B to A, or from A to B. Thus we may look upon $E-(V_B-V_A)$ or V_A-V_B+E as the effective E.M.F. from A to B. And if C is the current passing, and R the resistance of the wire, and we consider the battery to have an electrical resistance r of the same nature as the resistance of a wire, we get the equation

$$V_A-V_B+E=C(R+r).$$

Experiment shows that the results obtained from using this equation are consistent, so that we may always suppose a battery in a given condition to have a definite electrical resistance, just as any other part of a circuit has.

6. Simple Circuit with Battery.

Suppose the two ends A and B of the conducting wire are joined together, so that $V_A=V_B$. Then we have a battery with its two poles connected by means of a conducting wire. Let C denote the current flowing through the battery and wire, from the positive to the negative pole outside the battery; R and r being the wire and battery resistances. Thus we have the equation

$$E=C(R+r).$$

Or
$$C=\frac{E}{R+r}.$$

7. Kirchhoff's Laws. Suppose we have a net-work of linear conductors, that is, a system of linear conductors joined up together in any given manner, with given resistances, and including given E.M.F.s. And suppose the currents in the conductors to have attained their stationary values.

(1) Let us consider all the conductors which meet at one point of junction, and let the values of the currents flowing in the conductors towards that point be C_1, C_2, C_3,...C_n. Then since the quantity of electricity that flows up to the point per second is just equal to the quantity that flows away, we have

$$C_1+C_2+C_3+\ldots+C_n=0.$$

Or
$$\Sigma(C)=0\ldots. \qquad \text{(A)}$$

(2) Let us consider any complete circuit $ABC \dots KLA$ formed by some of the conductors of the system,

$$AB, BC, \dots KL, LA.$$

Let the resistances of these conductors be $R_1, R_2, \dots R_n$. Let them include E. M. F.s $E_1, E_2, \dots E_n$; and let there be currents flowing in them $C_1, C_2, \dots C_n$; the E. M. F.s and currents being both measured in the direction $ABC \dots KLA$.

Let the potentials of the points $A, B, C, \dots K, L$ be denoted by the letters $V_A, V_B, V_C, \dots V_K, V_L$. Then we have the equations

$$\begin{aligned} V_A - V_B + E_1 &= C_1 R_1, \\ V_B - V_C + E_2 &= C_2 R_2, \\ &\dots\dots\dots \\ V_K - V_L + E_{n-1} &= C_{n-1} R_{n-1}, \\ V_L - V_A + E_n &= C_n R_n. \end{aligned}$$

Adding these we obtain

$$E_1 + E_2 + \dots + E_n = C_1 R_1 + C_2 R_2 + \dots + C_n R_n.$$

Or $$\Sigma(E) = \Sigma(CR) \dots \qquad \text{(B)}$$

The results included in the formulae (A) and (B) are the laws given by Kirchhoff for a net-work of linear conductors.

8. Maxwell's Investigation for a System of Linear Conductors.

Suppose we have n points $A_1, A_2, A_3, \dots A_n$ connected together in pairs by $\frac{1}{2} n (n-1)$ conductors. Let the potentials of these points be $P_1, P_2, P_3, \dots P_n$. Let the conductivity (or reciprocal of the resistance) of the conductor joining A_p and A_q be K_{pq}. Let the internal E. M. F. (which may be zero) in this conductor acting from A_p to A_q be E_{pq}. And let the current from A_p to A_q along the conductor $A_p A_q$ be C_{pq}. Thus we have the relations

$$K_{pq} = K_{qp}, \ E_{pq} = -E_{qp}, \ C_{pq} = -C_{qp} \dots \qquad (1)$$

Also this system is not supposed to be necessarily complete in itself, as in the case to which Kirchhoff's Laws apply; but

suppose that currents Q_1, Q_2, Q_3,...Q_n enter it at the points A_1, A_2, A_3,...A_n. We must have

$$Q_1+Q_2+Q_3+\ldots+Q_n=0\ldots \tag{2}$$

Also we have the equations

$$C_{pq}=K_{pq}(P_p-P_q+E_{pq})\ldots \tag{3}$$

$$Q_p=C_{p1}+C_{p2}+C_{p3}+\ldots+C_{pn}\ldots \tag{4}$$

Thus

$$\begin{aligned}Q_p=&(K_{p1}+K_{p2}+K_{p3}+\ldots+K_{pn})P_p\\&-(K_{p1}P_1+K_{p2}P_2+K_{p3}P_3+\ldots+K_{pn}P_n)\\&+(K_{p1}E_{p1}+K_{p2}E_{p2}+K_{p3}E_{p3}+\ldots+K_{pn}E_{pn})\ldots\end{aligned} \tag{5}$$

The symbol K_{pp} does not occur in this equation. Let us suppose

$$K_{pp}=-(K_{p1}+K_{p2}+K_{p3}+\ldots+K_{pn})\ldots \tag{6}$$

Thus we get from (5)

$$\begin{aligned}&K_{p1}P_1+K_{p2}P_2+K_{p3}P_3+\ldots+K_{pp}P_p+\ldots+K_{pn}P_n\\&\qquad=K_{p1}E_{p1}+K_{p2}E_{p2}+\ldots+K_{pn}E_{pn}-Q_p\ldots\end{aligned} \tag{7}$$

We have in all n such equations as this among the n potentials $P_1, P_2,\ldots P_n$. These however are not all independent, for if we add the system of equations (8) the results on both sides will be identically zero from (1) (2) and (6). Thus there are $n-1$ independent equations from which we may determine $n-1$ differences of potential.

[We may notice that (7) is equivalent to

$$\begin{aligned}&K_{p1}(P_1-P_n)+K_{p2}(P_2-P_n)+\ldots+K_{p(n-1)}(P_{n-1}-P_n)\\&\qquad=K_{p1}E_{p1}+K_{p2}E_{p2}+\ldots+K_{pn}E_{pn}-Q_p.]\end{aligned}$$

Now let D denote the determinant

$$\begin{vmatrix}K_{11}, K_{12},\ldots K_{1(n-1)},\\ K_{21},\ \ _{22},\ldots K_{2(n-1)},\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ K_{n1}, K_{n2},\ldots K_{(n-1)(n-1)},\end{vmatrix}$$

which is symmetrical because $K_{pq}=K_{qp}$. And let D_{pq} denote the minor of K_{pq}. Then we have

$$\begin{aligned}(P_p-P_n)D=&(K_{12}E_{12}+\ldots+K_{1n}E_{1n}-Q_1)D_{p1}\\&+(K_{21}E_{21}+\ldots+K_{2n}E_{2n}-Q_2)D_{p2}+\ldots\end{aligned} \tag{8}$$

In the same way we may determine $P_q - P_n$, and so $P_p - P_q$, and thus find C_{pq}.

The coefficient of Q_q in $P_p - P_n$ is D_{pq}/D. This is equal to the coefficient of Q_p in $P_q - P_n$. For $D_{qp} = D_{pq}$.

It follows that if a current Q entering the system at A and leaving at B cause an E. M. F. P between C and D, a current Q entering at C and leaving at D will cause an E. M. F. P between A and B.

[For we have

$$D(P_C - P_D) = -Q(D_{AC} - D_{AD}) + Q(D_{BC} - D_{BD}),$$

and $$D(P_A - P_B) = -Q(D_{CA} - D_{CB}) + Q(D_{DA} - D_{DB}),$$

and the two expressions on the right-hand side are equal.]

If an E. M. F. E_{pq} act along the conductor A_{pq} the current in A_{rs} is

$$K_{rs} K_{pq} E_{pq} (D_{rp} + D_{sq} - D_{rq} - D_{sp}) \div D.$$

If there is no current in A_{rs}, the condition is

$$D_{rp} + D_{sq} - D_{rq} - D_{sp} = 0.$$

On account of the general relation $D_{pq} = D_{qp}$, this is easily seen to be the condition that there should be no current in A_{pq} when an E. M. F. acts in A_{rs}. When this relation holds the two conductors are called *conjugate*.

9. Conductors of any Form. Suppose we have a solid conductor of any form with two given electrodes, that is two surfaces in it, kept throughout at two given potentials V_1 and V_2 respectively. There will be a flow of electricity between the electrodes. And we can draw in the conductor a series of surfaces between the electrodes, each one of which is at the same potential throughout. These are called equipotential surfaces. We can draw a series of surfaces from one electrode to the other across which electricity never passes in either direction. These are called surfaces of flow. And the lines of intersection of these surfaces are called lines of flow.

Let dS be an element of an equipotential surface, ρ the specific resistance of the conductor at dS, dn an element of

normal to dS. Then the flow of electricity per second across dS is

$$-\frac{1}{\rho}\frac{dV}{dn}dS.$$

Now if dS' is any element of surface in the same position as dS, making an angle ϵ with it, and dn' is an element of normal to dS', the flow of electricity per second across dS' is

$$-\frac{1}{\rho}\frac{dV}{dn}dS'\cos\epsilon = -\frac{1}{\rho}\frac{dV}{dn'}dS'.$$

10. If we have two given states of potential and of flow in equilibrium, then the state got by superposing these two is in equilibrium also. For let V, V' be the potentials in two states at an element of surface dS. The sum of the two flows of electricity per second across dS in the two states is

$$-\frac{1}{\rho}\frac{d(V+V')}{dn}dS.$$

But $V+V'$ being the sum of the potentials at dS in the two states, this expression is what we should get for the flow across dS with the new distribution of potentials. Thus when the two given states are compounded, we have at every point a flow of electricity and a state of potential in equilibrium with each other. For neither would tend to alter unless the other were first altered. Thus the compounded state is a state in equilibrium.

11. The direction of the lines of flow in passing across the surface of separation of two media of different specific resistances is as a rule abruptly altered at the surface. Let the specific resistances of the two media be ρ_1, ρ_2. Let θ_1, θ_2 be the angles of inclination of a line of flow, at any point P on the surface of separation of the media, to the normal to the surface at that point. Let I_1, I_2 be the currents per unit area at P in the two media normal to the surface of separation; I'_1, I'_2 the currents per unit area parallel to the surface of separation. Let dn be an element of normal to the surface of separation of the media at P measured in the same direction in both media,

towards the surface in the first, away from it in the second. Let dl be an element of length of the surface of separation measured away from P, in the direction in which the potential decreases most rapidly. Then I'_1, I'_2 are both in the direction of dl, that is the elementary portions of a line of flow on the two sides of P are in the same plane with the normal to the surface of separation at P.

And $$\tan\theta_1 = \frac{I'_1}{I_1},\ \tan\theta_2 = \frac{I'_2}{I_2}.$$

$$I'_1 = -\frac{1}{\rho_1}\frac{dV}{dl},\ I'_2 = -\frac{1}{\rho_2}\frac{dV}{dl}.$$

And we must have $I_1 = I_2$, for as much electricity flows away from the surface of separation per second as flows up to it.

Therefore $$\rho_1 \tan\theta_1 = \rho_2 \tan\theta_2.$$

12. In any conductor let c be the specific conductivity, that is the reciprocal of the specific resistance, and V the potential, at any point. Consider a rectangular parallelepiped $dx\,dy\,dz$ in the conductor, having two opposite corners at the points (x, y, z) $(x+dx, y+dy, z+dz)$. The current flowing into the parallelepiped across the face $dy\,dz$ is

$$-dy\,dz\,c\frac{dV}{dx}.$$

That which flows out at the opposite face is

$$-dy\,dz\left\{c\frac{dV}{dx}+\frac{d}{dx}\left(c\frac{dV}{dx}\right)dx\right\}.$$

So that the gain of electricity per second in the parallelepiped, due to these two faces, is

$$dx\,dy\,dz\frac{d}{dx}\left(c\frac{dV}{dx}\right).$$

And we have similar expressions for the other two pairs of faces. But when the permanent state of flow has been attained there is no gain or loss of electricity in the given elementary parallelepiped. Thus we have the equation

$$\frac{d}{dx}\left(c\frac{dV}{dx}\right)+\frac{d}{dy}\left(c\frac{dV}{dy}\right)+\frac{d}{dz}\left(c\frac{dV}{dz}\right)=0.$$

Or if the conductor is homogeneous, we have

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0.$$

13. Anisotropic Medium. There are certain media which have different specific conductivities in different directions. In any such medium there are always three directions at right angles to each other such that an E. M. F. in one of these directions produces a current entirely in that direction. These three directions are called principal axes of conductivity. Let us take them as axes of coordinates, and let the specific conductivities in the directions of them be c_1, c_2, c_3. Let dS be an element of an equipotential surface; dn an element of the normal to dS. Let the current at dS be I, and the components of I parallel to the axes, I_1, I_2, I_3. Let the direction cosines of dn be l, m, n; and those of I, l', m', n'. Then we have

$$I_1 = -c_1 l \frac{dV}{dn}, \quad I_2 = -c_2 m \frac{dV}{dn}, \quad I_3 = -c_3 n \frac{dV}{dn}.$$

Therefore
$$\frac{l'}{c_1 l} = \frac{m'}{c_2 m} = \frac{n'}{c_3 n}.$$

Thus the current will not be in the same direction as the E. M. F. when c_1, c_2, c_3 are different unless two of the quantities l, m, n are zero. That is, the E. M. F. must be in the direction of a principal axis.

14. Suppose we have a conductor formed of an indefinitely extended thin sheet of a conducting substance, of thickness t, and specific resistance ρ; and let there be two electrodes consisting of small cylindrical surfaces with axes perpendicular to the sheet, of radius a, and with their axes a distance apart b, which is large compared with a. To find the resistance of the conductor. Let I be the current flowing from one electrode to the other; V_1, V_2 the potentials of the electrodes.

Now if we consider a current I to flow away from the positive electrode to an infinite distance, the equipotential surfaces will be cylinders coaxal with the electrode, and the lines of flow

straight lines meeting its axis at right angles. If V is the potential at a distance r from the axis, we have

$$I = -\frac{2\pi r t}{\rho}\frac{dV}{dr};$$

$$\therefore \quad V = C - \frac{I\rho}{2\pi t}\log r,$$

by integrating, C being a constant.

Thus the potential difference between the positive electrode and the axis of the negative, that is practically between the electrodes, due to the current I from the positive one is

$$\frac{I\rho}{2\pi t}\log\frac{b}{a}.$$

And we have also the same potential difference in consequence of the current $-I$ from the negative electrode. Thus we have

$$V_1 - V_2 = \frac{I\rho}{\pi t}\log\frac{b}{a}.$$

And the approximate value of the resistance is

$$\frac{\rho}{\pi t}\log\frac{b}{a}.$$

Let us consider the sheet to be indefinitely thin, and the electrodes to be points on it. The potential at any point on the sheet at distances r_1, r_2 from the electrodes will be

$$\frac{I\rho}{2\pi t}\log\frac{r_2}{r_1} + \text{a constant.}$$

Thus the lines of equal potential will be lines for which r_2/r_1 is constant. These are a system of coaxal circles of which the electrodes are the limiting points. The lines of flow will be circles orthogonal to these or circles through the electrodes. If we made the electrodes of any size, and circular, they would become two of the circles of the coaxal system which are the equipotentials.

15. Suppose we have an indefinitely extended conducting medium of specific resistance ρ. To find the resistance between two

small spherical electrodes of radius a, placed in the medium. Let I be the current flowing between the electrodes, whose potentials are V_1 and V_2. Let V be the potential at a distance r from the centre of the positive electrode, due to the current I flowing from this electrode. Then we have

$$I = -\frac{4\pi r^2}{\rho}\frac{dV}{dr}.$$

$$\therefore \quad V = \frac{I\rho}{4\pi r},$$

since the potential vanishes at an infinite distance from the electrode.

Thus $$V_1 = \frac{I\rho}{4\pi a},$$

neglecting the influence of the current $-I$ at the negative electrode on the positive one.

Similarly $$V_2 = -\frac{I\rho}{4\pi a}.$$

$$\therefore \quad V_1 - V_2 = \frac{I\rho}{2\pi a}.$$

Therefore the resistance is

$$\frac{\rho}{2\pi a},$$

a value independent of the distance between the electrodes.

The resistance in a conductor bounded on one side by a plane, but otherwise indefinitely extended, between two hemispherical electrodes of radius a is double of this result, that is

$$= \frac{\rho}{\pi a}.$$

This is practically the case of the conduction between two electrodes in the earth, at places not too far apart.

16. Suppose we have a condenser of which K is the specific inductive capacity of the dielectric, C the capacity, $Q, -Q$ the charges on the conductors, ρ the specific resistance of the

dielectric, R the resistance of the dielectric considered as an electrical conductor with the condenser conductors as electrodes, I the entire current in this conductor.

Then if V denotes potential, n the normal to the positively charged conductor measured into the dielectric, dS an element of the positively charged surface, we have

$$dI = -\frac{1}{\rho}\,dS\,\frac{dV}{dn},$$

$$I = -\frac{1}{\rho}\iint\frac{dV}{dn}\,dS.$$

And if σ is the surface density at dS,

$$4\pi\sigma = -K\frac{dV}{dn},$$

$$\therefore\quad Q = \iint\sigma\,dS = -\frac{K}{4\pi}\iint\frac{dV}{dn}\,dS.$$

$$\therefore\quad \rho I = \frac{4\pi Q}{K}.$$

Now if V_1, V_2 are the potentials at the positively and negatively charged conductors, we have

$$V_1 - V_2 = RI = \frac{Q}{C}.$$

$$\therefore\quad \frac{\rho}{R} = \frac{4\pi C}{K}.$$

17. Law of Joule. Consider a portion of a circuit of resistance R, in which a steady current C is flowing. The E. M. F. of the extremities of this portion is CR. So that the work done in it per second in joules is C^2R. This energy is entirely expended in generating heat in the conductor; so that C^2R is the mechanical measure, or the measure in joules, of the quantity of heat developed in the conductor per second.

If H is the number of calories of heat developed in the

conductor in a given time, say t seconds; and J is the mechanical equivalent of heat, we have

$$JH = C^2 Rt.$$

This relation is Joule's Law for the quantity of heat developed in a conductor by an electric current. It has been verified experimentally by Joule.

CHAPTER III.

Mechanical and Electrical Units.

1. Mechanical Units. Before speaking of electrical units we shall consider a few mechanical units which are of great importance in electrical work.

The *fundamental units* required in mechanics and electricity are those of *length*, *mass*, and *time*. Units of all other quantities, that we have to consider, are derived from these and are called *derived units*.

The fundamental units used by all electricians are the *centimetre*, *gramme*, and *second*: and the system of units, mechanical and electrical, derived from these is called the *centimetre-gramme-second system*, or *C. G. S. system*.

Two of the derived mechanical units of the C. G. S. system we shall find of great importance. These are the units of force and of work, called the *dyne* and the *erg*. We give their formal definitions.

Def.: **THE DYNE is the force which acting on a mass of one gramme for one second generates it in a velocity of one centimetre per second.**

Def.: **THE ERG is the work done by one dyne when it moves its point of application through a distance of one centimetre in the direction in which it acts.**

2. As British Engineers use quite a different system of units from this, we shall compare the principal units in the two systems.

The fundamental units of the British Engineers' system are the *foot*, the *pound*, and the *second*.

The unit of work, called the *foot-pound*, is not derived in the same way as the erg, but its definition is as follows.

DEF.: **THE FOOT-POUND is the work done when a force equal to the weight of a pound moves its point of application through a distance of one foot in the direction in which it acts.**

Thus this unit is not a perfectly constant one, like the erg, but depends on the value of acceleration due to gravity in any given place.

To compare the two systems, we have the following relations.

$$1 \text{ centimetre} = \cdot 0328 \text{ feet},$$
$$1 \text{ gramme} = \cdot 002204 \text{ pounds};$$

the unit of time is the same in both systems.

The acceleration due to gravity is in the British Engineers' system about 32·2 units of acceleration, or 32·2 feet per second per second.

Now $32\cdot 2 \text{ feet} = 32\cdot 2 \div \cdot 0328 \text{ cms.} = 981 \text{ cms.}$

Thus the acceleration due to gravity is 981 units of acceleration in the C. G. S. system.

That is, the unit of mass is acted on by the earth's attraction with 981 units of force, or the weight of a gramme is 981 dynes.

To compare the foot-pound and the erg, we notice that

$$1 \text{ foot} = \frac{1}{\cdot 0328} \text{ cms.};$$

$$1 \text{ pound's weight} = \frac{981}{\cdot 002204} \text{ dynes.}$$

$$\text{Thus a foot-pound} = \frac{981}{\cdot 002204 \times \cdot 0328} \text{ ergs}$$
$$= 13{,}833 \times 981 \text{ ergs} = 1\cdot 356 \times 10^7 \text{ ergs.}$$

To find the value of a foot-pound in terms of an erg, we must always multiply 13,833 by the number of units of acceleration due to gravity expressed in the C. G. S. system for the given locality. Taking 981 as the approximate value of this, we get the number $1\cdot 356 \times 10^7$.

The erg as a unit of work is inconveniently small, so that instead of this as a practical unit *joule*, equal to 10^7 ergs, is used.

Thus the foot-pound = 1·356 joules.

The practical unit of power, or rate of doing work, adopted is the *watt*, which is equal to one joule per second.

The British Engineers' unit of power is the *Horse-Power*, which is the rate of doing work at 33,000 foot-pounds per minute, or 550 foot-pounds per second.

$$\begin{aligned}\text{One Horse-Power} &= 550 \times 1{\cdot}356 \text{ watts}\\ &= 745{\cdot}8 \text{ watts}\\ &\text{or } 745{\cdot}8 \text{ joules per second.}\end{aligned}$$

3. The electrical units which we shall now obtain are based on the magnetic actions of the electric current, and are called Electro-Magnetic Units to distinguish them from the Electro-Static units already obtained, which are based on the electro-static actions of electrical charges.

4. Electrical Units. The electric units, as we have said, are derived from the C. G. S. system of units: but the electrical units derived directly from this system are found to be of inconvenient size for practical use; so that other *practical units*, obtained by multiplying these by integral powers of 10, are adopted.

Unit of Current. The C. G. S. unit of current has been already defined with reference to the magnetic action of the current as follows.

DEF.: **THE UNIT OF CURRENT is the current which flowing in a conductor in the form of a circular arc one centimetre long, and having a radius of one centimetre, acts with a force of one dyne on a unit magnetic pole placed at the centre of the circle.**

The practical unit of current is the AMPERE, which is $\frac{1}{10}$ of the C. G. S. unit of current.

DEF.: **THE UNIT OF ELECTRICITY is the quantity which passes between two points in one second when unit current is flowing between them.**

The practical unit of electricity is the COULOMB, and defined with reference to the ampere it is $\frac{1}{10}$ of the C. G. S. unit.

5. Unit of Electromotive Force. The E. M. F. or P. D. between two points is the work done by the electrical forces on a unit of electricity passing from the point of higher to the point of lower potential; that is, it is the work done per second on every unit of current passing between the two points. Thus we have the definition:

DEF.: **THE UNIT OF E. M. F. is the E. M. F. that must exist between two points when one erg of work is done per second for every unit of current that passes between the two points.**

The practical unit of E. M. F. is the VOLT, which is the E. M. F. that must exist between two points when a joule of work is done per second for every ampere passing between the two points.

If in any system of units E is the E. M. F. between two points, C the current flowing between them, and W the work done per second between them, we have the equation

$$W = EC.$$

To find the relation between the volt and the C. G. S. unit of E. M. F. let us suppose the E. M. F. between two points to be a volt, and let a current of an ampere be flowing between them. Then we know that a joule of work is being done per second between them, or work is being done at the rate of a watt.

Now let us write the equation

$$W = EC$$

as an equation in C. G. S. units, for the work done per second between the two points; and express all the quantities in C. G. S. units. We know that $W = 10^7$ C. G. S. units of power, and $C = 10^{-1}$ C. G. S. units of current.

Thus writing the above equation in C. G. S. units, we have

$$10^7 = E \cdot 10^{-1}$$
$$\therefore\ E = 10^8.$$

Thus the volt contains 10^8 C.G.S. units of E.M.F.

6. Unit of Resistance. We have seen by Ohm's Law and the definition of resistance that, using any units of E. M. F., current, and resistance, if we have a steady current C flowing through a conductor of resistance R, its terminals being at P. D. E, then we have the relation

$$E = CR.$$

From this we get the definition of the unit of resistance.

DEF.: **THE UNIT OF RESISTANCE is the resistance of a conductor, such that when unit of current flows steadily through it, its terminals are at unit of potential difference.**

The practical unit of resistance is the OHM, and is the resistance of a conductor such that when an ampere flows steadily through it its terminals are at P. D. of a volt.

To find the relation between the ohm and the C. G. S. unit of resistance, suppose we have a conductor whose resistance is an ohm, and that a current of an ampere is flowing steadily through it. Its terminals are at a P. D. of a volt.

Let us write the equation

$$E = CR$$

as an equation in C. G. S. units for the P. D. of the terminals of the given conductor, and express all the quantities in C. G. S. units. We have $E = 10^8$ C. G. S. units of E. M. F., and $C = 10^{-1}$ units of current.

Thus $$10^8 = 10^{-1} . R.$$

$$\therefore \quad R = 10^9.$$

Thus the ohm contains 10^9 C. G. S. units of resistance.

7. Unit of Capacity. If we have a condenser of capacity K, and by charging it with a quantity Q of electricity we raise its terminals to P. D. E, then K, Q, and E being expressed in any consistent units, we have the relation

$$Q = KE.$$

From this we get the definition of the unit of capacity.

DEF. : **THE UNIT OF CAPACITY is the capacity of a condenser which is raised to unit of potential difference by a charge of unit quantity of electricity.**

The practical unit of capacity is the FARAD, which is the capacity of a condenser that would be charged to a P.D. of one volt by a charge of one coulomb of electricity.

The farad however is an inconveniently large unit, and for convenience capacities are usually measured in *microfarads*, a microfarad being $\frac{1}{1,000,000}$ of a farad.

To find the relation between the farad and the C. G. S. unit of capacity, suppose we have a condenser of a farad capacity charged with a coulomb of electricity. It is at a P. D. of a volt.

Let us write the equation

$$Q = KE$$

as an equation in C. G. S. units for the quantity of electricity with which the condenser is charged, and express all the quantities in C. G. S. units. We have $Q = 10^{-1}$ of a C. G. S. unit of electricity, $E = 10^8$ C. G. S. units of P. D.

Thus
$$10^{-1} = K . 10^8.$$
$$\therefore \quad K = 10^{-9}.$$

Thus the farad is 10^{-9} of the C. G. S. unit of capacity.

And the microfarad $= 10^{-15}$ of the C. G. S. unit of capacity.

CHAPTER IV.

MEASUREMENT OF RESISTANCE.

1. WE have seen that the practical unit of resistance is the ohm. To measure the electrical resistance of a conductor, then, is to determine by what number its resistance, as expressed in ohms, is to be denoted.

Methods of finding the value of the ohm, or of finding the value in ohms of the resistance of a given conductor, based on the principles by which the ohm has been defined, will be spoken of later on. Such a process is called measuring the resistance of a conductor in absolute measure.

In the ordinary process of finding the resistance of a conductor it is found by comparison with those of given conductors which have previously been determined from experiments on absolute measure of resistances.

The unit of resistance, from which many standard resistance coils at present in use have been made, was determined by a committee of the British Association from experiments made by them in 1863. It was intended to be the true ohm; but, as the value of this has since been found with much greater accuracy, it is best called the *B.A. unit of resistance.* It is the resistance of a column of mercury 104·83 cms. long and 1 sq. mm. cross-section at temperature 0° *C.*

From more recent determinations the Paris Congress of 1884 adopted, provisionally, as the length of the mercury column with cross-section and temperature as above, whose resistance should be the standard of resistance, the value 106 cms. This was called the *legal ohm.*

A still more accurate value was adopted by the British Association committee on Electrical Standards in 1892. The length of the column taken by them is 106·3 cms.

2. To measure a resistance then we make use of a given standard set of coils, whose resistances are known in terms of some definite unit of resistance (say the B. A. unit), and we have to determine how many times the given resistance contains the unit. This process is also called *comparison of resistances.*

A set of resistance coils, so arranged that we may obtain from it by suitable adjustments a resistance of any number of units required up to a certain limit, forms an arrangement called a *resistance box.*

3. A few points in the practical construction of resistance coils should be noticed.

The coils are either arranged so that they may be immersed in water at a standard temperature, at which their resistances have been adjusted; or their resistances are corrected to allow for variations of temperature.

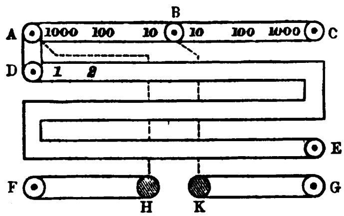
The wire of which the coils are to be made is covered with insulating material, and before winding, it is doubled on itself, so that throughout the coil there are equal and opposite currents flowing side by side. The object of this is (as we shall see later on) to avoid induction effects.

The ends of each coil are connected to two points in a thick strip of brass which is bored with a conical hole between the ends of the coil, and into this hole a conical plug fits. The strip is then slit across through the hole; so that when the plug is inserted the resistance along the strip from one end of the coil to the other is practically nil, but when the plug is removed we have the entire resistance of the coil. An entire set of coils is connected up in series in this way along a strip of brass, and between the ends of each coil there is a gap with a conical hole and a plug fitting into it. We can insert in the series the resistance of any coil by removing the plug corresponding to it.

Resistance boxes are usually made with their coils of resistances

equal to the following multiples of the unit: 1, 2, 3, 4, 10, 20, 30, 40, 100, ... With these we can easily obtain any multiple of the unit up to the sum of all the resistances of the box.

4. A special form of resistance box, the use of which we shall presently explain, is the Post Office Box. A diagram of the resistances and connexions of this box is shown in the figure. The double lines represent the thick strips of brass fastened down to an ebonite slab. The numbers denote the resistances of coils lying below the ebonite slab, and inserted along the strips of brass in the way already explained. At $A, B, C, D, E, F, G,$ are binding screws. FH and KG are flexible pieces of brass having the ends F and G fixed; and at the other ends H and K are ebonite knobs, by pressing which down contact is made by platinum contacts, thus putting F and G in connexion with A and B respectively.



5. Substitution Method. This is the simplest way of measuring the resistance of a coil. A constant battery, a galvanometer, and an adjustable set of resistances are required. The coil is joined up in series with the battery and the galvanometer, and the deflexion of the galvanometer observed. The coil is then removed and replaced by the resistance box, the resistance of which is adjusted till the same deflexion in the galvanometer is obtained. Thus we have the same current passing through the galvanometer, and the entire resistance of the circuit must be the same. Therefore the resistance of the coil must be equal to that to which the resistance box has been adjusted.

Supposing the E. M. F. and resistance of the battery and the resistances of all other parts of the circuit to remain constant, the degree of accuracy to which we can measure the resistance of a coil by this method will depend upon two things:

(i) The accuracy to which we can adjust the substituted resistance to any given value, or the accuracy with which we can read off its value when adjusted;

(ii) The sensitiveness of the galvanometer at the deflexion at which we use it.

In using this method we may suppose that all the error arises from want of sensitiveness in the galvanometer, because it would be found, as a rule, that on varying the resistance of the circuit by amounts which can easily be measured, the variation of the galvanometer deflexion is inappreciable.

Suppose we use a given tangent-galvanometer having a reduction factor K, and a battery of E. M. F. E. Let the resistance to be measured be R, and the resistance of the remainder of the circuit r.

If θ is the deflexion of the galvanometer,

$$\frac{E}{R+r} = K \tan \theta.$$

Suppose the substituted resistance is adjusted to a value $R+dR$, differing slightly from R, and let this produce a deflexion $\theta + d\theta$ in the galvanometer. Then supposing a small variation of the galvanometer deflexion to be equally appreciable all over the scale of the galvanometer, for greatest accuracy we must adjust matters so that a given variation of resistance dR produces the greatest possible variation of deflexion $d\theta$.

Now suppose we can vary E without much altering the value of r, which will be the case if we have battery cells of small resistance compared with that of the galvanometer.

We have from the above equation

$$R+r = \frac{E}{K}\cot\theta,$$

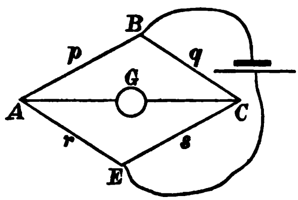
$$\therefore \quad dR = -\frac{E}{K}\operatorname{cosec}^2\theta \,.\, d\theta,$$

$$\therefore \quad d\theta = -\frac{K}{E}\sin^2\theta \,.\, dR,$$

$$= -\frac{dR}{R+r}\sin\theta\cos\theta.$$

Thus for the variation dR to produce as great a variation $d\theta$ in the deflexion as possible we must make r as small as is convenient, that is, introduce no unnecessary resistance into the circuit, and then adjust E so that $\sin\theta\cos\theta$ is as great as possible, that is, have a deflexion as near to 45° as possible.

6. Wheatstone's Bridge. This is an arrangement for measuring resistances constructed in the manner indicated in the diagram. BC, CE, BA, AE, are four conductors. The poles of a battery are connected to B and E, and the terminals of a galvanometer to A and C. The resistances of the conductors are so arranged that there is no current in the galvanometer.



Now let the resistances of the conductors be, as denoted in the figure, p, q, r, s. And let the potentials of A, B, C, E, be V_A, V_B, V_C, V_E. Since there is no current in the galvanometer, $V_A = V_C$.

Since the same current flows along BAE we have

$$\frac{V_B - V_A}{V_B - V_E} = \frac{p}{p+r}.$$

And since the same current flows along BCE we have

$$\frac{V_B - V_C}{V_B - V_E} = \frac{q}{q+s},$$

$$\therefore \quad \frac{p}{p+r} = \frac{q}{q+s},$$

$$\therefore \quad \frac{p}{r} = \frac{q}{s}.$$

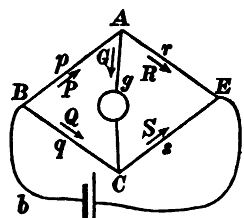
If then we know three of the resistances p, q, r, s, we can find the fourth.

In the working of this method a contact-key is used in the battery circuit and one in the galvanometer circuit, so that the battery and galvanometer may be put in for a short while at a time, and not kept on all the time that the adjustments are being made.

The Post Office resistance box is a very convenient arrangement for making this test. The battery has its poles connected to the screws E and G, and the galvanometer its terminals to C and F. The resistance to be measured is joined up from C to E. When the knob K is depressed the battery is put on from B to E. When H is depressed the galvanometer is put on from A to C. Thus the arms CB, BA, AE, and the resistance CE, take the place of q, p, r, s in the diagram; the letters A, B, C, E occupying corresponding places in the two figures.

In making this test the battery-key should be always depressed before the galvanometer-key, to avoid induction effects in the galvanometer, as we shall see later on.

7. Let us consider what the currents in the various arms will be with any given resistances. Suppose we have a battery of E. M. F. E, and let the resistance of the battery circuit from B to E be b, and that of the galvanometer circuit from A to C be g. Let the currents flowing through the various resistances be denoted by the corresponding capital letters, as in the figure.



We have at once the relations, obtained from Kirchoff's first law,

$$B = P + Q, R = P - G, S = Q + G.$$

So that it is only necessary to obtain three more independent equations in P, Q. G. These we get by applying Kirchhoff's second law to the three circuits BAC, CAE, BCE. Thus we have

$$\begin{aligned} Gg + Pp - Qq &= 0, \\ -Gg + (P - G)\,r - (Q + G)\,s &= 0, \\ (P + Q)\,b + Qq + (Q + G)\,s &= E. \end{aligned}$$

These we may rewrite as

$$\begin{aligned} Gg + Pp - Qq &= 0, \\ -G(g + r + s) + Pr - Qs &= 0, \\ Gs + Pb + Q\,(b + q + s) &= E. \end{aligned}$$

Let us write D for the determinant

$$\begin{vmatrix} g, & p, & -q, \\ -(g+r+s), & r, & -s, \\ s, & b, & b+q+s. \end{vmatrix}$$

Then for the value of G, the galvanometer current, we have

$$G=\frac{E(qr-ps)}{D}.$$

This again shows what relation must exist among the resistances that there may be no current in the galvanometer.

Similarly the currents in the other arms may be found.

8. Conditions of Accuracy. For accuracy in determining the value of an unknown resistance by this method we must arrange so that an error made in the estimated value of the unknown resistance may be easily detected by a deflexion of the galvanometer.

Suppose that p is the resistance to be measured, and q, r, s are adjustable. In measuring the resistance of a conductor only a very small current must be passed through it; for the effect of a current is to raise its temperature, and so alter its resistance, so that a large current passed through the conductor will produce an error in the result. Suppose then we fix the value of the current P which is to pass through the resistance p. The value of the galvanometer current in terms of this, found from the first two of the equations given above, by eliminating Q, is

$$G=\frac{P\,(rq-ps)}{gs+gq+rq+sq}.$$

It may be noticed that this is independent of the E. M. F. and resistance of the battery used, so long as the value of P is determined.

After having made the adjustments as well as possible the estimated value of p is rq/s. Now suppose this is not the true value, but that an error dp is made in the measure of p, so that $rq/s=p+dp$. Thus we get

$$G=\frac{P\,s\,dp}{gs+gq+rq+sq}.$$

Now the deflexion of the galvanometer will depend on its construction and on G. Suppose we are given the space for the galvanometer-coils, and we wish to determine what wire it must be wound with so as to be suitable for making the given measurement. If we have the space wound with wire of a certain gauge, and we alter the cross-section of the wire in the ratio $1 : n$, we also alter its length in the ratio $n : 1$, and (supposing the thickness of the insulation to bear a constant ratio to the diameter of the wire) we alter its resistance in the ratio $n^2 : 1$. But the sensitiveness of the galvanometer, or the deflexion produced by a very small current, is proportional to the number of turns; it is therefore altered in the ratio $n : 1$. Thus the sensitiveness is proportional to $\sqrt{g}$. Therefore the deflexion produced is proportional to

$$\frac{P.s\,dp\sqrt{g}}{g(s+q)+q(r+s)}.$$

This is a maximum for variations of g when

$$g=\frac{q(r+s)}{s+q}, \quad \text{or} \quad =\frac{r(p+q)}{p+r}.$$

This then is the best value for the galvanometer resistance.

The winding of the galvanometer would not, as a rule, be made to have this value, but this shows us that if the arms are of high resistance it is best to use a galvanometer wound with fine wire; and if they are of low resistance, one wound with thick wire.

If we arrange to give the above value to the galvanometer resistance we get

$$G=\frac{Ps\,dp}{2q(r+s)}=\frac{P\,dp}{2(p+q)}.$$

And the deflexion is proportional to

$$\frac{P\,dp\sqrt{r}}{\sqrt{(p+q)(p+r)}}.$$

From this it follows that, as far as this consideration alone is concerned, we should make q very small and r very large.

9. But the accuracy of the determination will depend upon two distinct considerations,

(i) The sensitiveness and suitability of the galvanometer;

(ii) The accuracy to which the adjustable resistances can be adjusted and read off.

To consider the second of these; since the estimated value of p is given by

$$p=\frac{qr}{s},$$

we see that, if dq, dr, ds are errors made in reading the values of q, r, s, and dp is the corresponding error in the estimated value of p, supposing now that there is absolutely no current in the galvanometer,

$$\frac{dp}{p}=\frac{dq}{q}+\frac{dr}{r}-\frac{ds}{s}.$$

Therefore any *relative error* made in q, r, or s will produce an equal relative error in p. So that if we wish to measure p to the same *absolute* accuracy as q, r, and s can be adjusted to, none of these should be smaller than p.

Again, it is not well to make the resistance r very high on account of the error due to heating that the current which passes through p would produce in it.

If the four resistances p, q, r, s are taken to be equal, the best value for the galvanometer resistance is the same.

10. We have seen that when the balance has not been completely obtained the current through the galvanometer is

$$G=\frac{E(qr-ps)}{D};$$

and D is equal to

$$bg(p+q+r+s)+b(p+q)(r+s)+g(p+r)(q+s)$$
$$+pr(q+s)+qs(p+r).$$

Now if we interchange the battery and galvanometer we get for the current through the galvanometer

$$G'=\frac{E(qr-ps)}{D'};$$

where D' is another determinant function of the resistances, whose value, it can easily be seen, is got from that of D by interchanging the letters g and b. Thus G is greater or less than G' according as D' is greater or less than D.

But
$$D-D' = (b-g)\{(p+q)(r+s)-(p+r)(q+s)\}$$
$$= (b-g)(p-s)(r-q).$$

Suppose that b is greater than g. And suppose that p and r are both greater than, or both less than, s and q respectively. Then D is greater than D'. Thus the second arrangement is more sensitive.

In the same way by examining all the cases we see that the following rule holds:

If of the two resistances, that of the battery and that of the galvanometer, the greater connects the junction of the two highest with that of the two lowest resistances of the four arms, the deflexion in the galvanometer is the greatest.

11. Measurement of a resistance containing an E. M. F.

Suppose in the arm BA whose resistance is to be measured there is an E. M. F., e. Our equations must now be written

$$Gg+Pp-Qq=e,$$
$$-G(g+r+s)+Pr-Qs=0,$$
$$Gs+Pb+Q(b+q+s)=E.$$

The value of G obtained from these is

$$G=\frac{E(qr-ps)+e\{b(r+s)+r(q+s)\}}{D}.$$

Now suppose the balance obtained, that is $qr=ps$, then the value of G is independent of E. Also we have

$$D=g\{b(p+q+r+s)+(p+r)(q+s)\}+b(p+q)(r+s)$$
$$+r(p+q)(q+s)-q(qr-ps).$$

That is, in this case,

$$D=\frac{g}{s}\left\{b(r+s)(q+s)+r(q+s)^2\right\}+\frac{bq}{s}(r+s)^2$$
$$+\frac{qr}{s}(r+s)(q+s).$$

Thus $$G = \frac{e.s}{g(q+s)+q(r+s)},$$

which is also independent of b.

To test whether the balance is obtained in this case, the course is to make the galvanometer circuit first, which will produce a deflexion of the galvanometer, and to see that no further permanent deflexion is produced on *then* making the battery circuit. A throw of the galvanometer needle, due to self-induction, may be obtained on making the battery circuit: but if the same *permanent* deflexion is produced in the two cases, that is, with battery-key up and down, then the balance has been obtained.

12. Mance's Test for Battery Resistance.

This is a particular application of the above. The battery whose resistance is to be tested is inserted in an arm of the bridge. The testing battery is dispensed with, but the points B and E may be connected through a contact-key. The galvanometer deflexion, with B and E unconnected, is first obtained, and then if on connecting B and E this is unchanged, the balance has been obtained.

If the battery to be tested is of an inconveniently high E. M. F., consisting of several cells, it may be joined up so as to have the same resistance but a low E. M. F. by making some of the cells oppose the others.

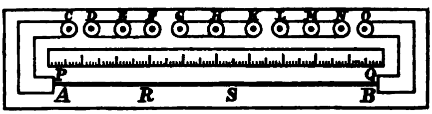
13. Suppose we wish to obtain the balance with E. M. F.s in any number, say in all, of the arms. Let these be e_1, e_2, e_3, e_4 in p, q, r, s respectively. Then our equations become

$$\begin{aligned}
&Gg &&+Pp-Qq &&= e_1-e_2,\\
-&G(g+r+s) &&+Pr-Qs &&= e_3-e_4,\\
&Gs &&+Pb+Q(b+q+s) &&= E+e_2+e_4.
\end{aligned}$$

Solving for G by determinants we find that its value consists of five terms having as factors E, e_1, e_2, e_3, e_4; the term in E disappearing when the balance has been obtained. And in the same way as before it follows that the terms in e_1, e_2, e_3, e_4 are independent of b when the balance has been obtained: so that the test for the balance is performed as

before, that is, by observing that the permanent deflexion of the galvanometer is not altered on putting in the testing battery by means of its key.

14. The Metre Bridge. This is another piece of apparatus for realizing Wheatstone's Bridge in practice. AB is a wire a metre in length, which we will suppose to be of uniform resistance throughout, made of some material such as platinoid, an alloy of German silver and tungsten, which has a high specific resistance, and changes very little in resistance for variations of temperature. This is soldered to a strip of thick copper $ACD...B$, in which are four gaps at CD, FG, KL, NO, and eleven binding screws $C, D, E, F, G, H, K, L, M, N, O$. Over AB slides a contact-piece, by pressing a knob of which contact can be made with any point in AB, and the distance through which this has been moved is measured by means of the scale PQ divided into millimetres.



15. The following are two of the principal uses of the bridge.

(1). **To compare the resistance of two coils.**

Insert the coils in the gaps FG, KL, connecting their ends to the binding screws F, G, K, L. Connect CD, NO by short stout pieces of copper. We may now suppose these connectors and all the copper strips of the bridge to have no resistance. Connect the poles of a testing battery to E and M, and the terminals of a galvanometer to H and the contact-piece. Adjust this piece till no deflexion is obtained in the galvanometer on making contact with the wire. Suppose R is the point on the wire so obtained. Then the ratio of the resistances at FG and KL is equal to that of the resistances AR and RB, or of the length AR and RB as given on the divided scale.

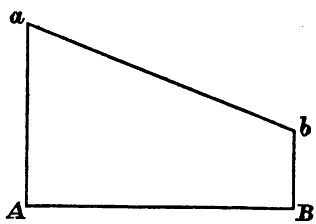
(2). **To determine the difference between two nearly equal resistances in terms of the length of a portion of the bridge wire.**

Join up battery and galvanometer as before. Put the two

resistances p and q in CD and NO, and two auxiliary resistances nearly equal to each other, and as nearly as convenient equal to the given resistances in FG and KL. Find the point for the contact piece at which the balance is obtained. Let this be R. Now interchange p and q and find a new position S for the contact piece. The entire resistance $EABM$ is the same in the two cases, and it is divided at the points R and S in the same ratio in the two cases. Thus the resistance of EAR through p must be the same as that of $EARS$ through q. Thus we have $p-q=$ resistance of RS.

It should be noticed that in this case resistances in the copper strips will make no error in the result.

16. Graphic Representation of Ohm's Law. Suppose we have a linear conductor in which a current is flowing. Take a straight line AB as a base line, and let equal lengths along it represent parts of equal resistance along the given conductor. Let us erect ordinates Aa, Bb, &c. to represent the potentials at the corresponding points of the conductor. Then since with a given current the fall of potential between two points is proportional to the

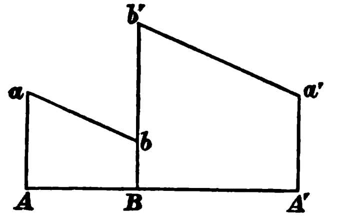


resistance between those two points, the locus determined by the ends of these ordinates is a straight line inclined to AB. And if we take distances along AB to represent resistances, according to any scale, in ohms, and the lengths of the ordinates to represent potentials, or potential-differences, according to the same scale, in volts, the current flowing in the given conductor will be represented in amperes by the tangent of the angle between AB and ab.

17. If a sudden variation of potential occurs at any point of the conductor, this will be represented by a part of the line ab being displaced parallel to itself, by a distance, measured vertically, which represents the number of volts in the given

potential-difference. In such a diagram, since the same current passes through the various parts of the conductor, the various parts of *ab* will be inclined at the same angle to the base line.

Suppose we have a closed electric circuit containing a battery, so that a current is flowing in the circuit. We may, for shortness, consider the battery to cause a sudden variation of the potential at one point of the circuit, this being equal to the E. M. F. of the battery. Take AA' to represent the resistance of the whole circuit, so that A and A' denote the same point, and the ordinates Aa, $A'a'$ are equal. Let B be the point at which the change of potential due to the battery is supposed to occur. Let bb' denote this change of potential. The variations of potential round the circuit will be represented as in the figure, in which the first and last points of the line of potentials $ab\,b'a'$ must always be on the same horizontal level.

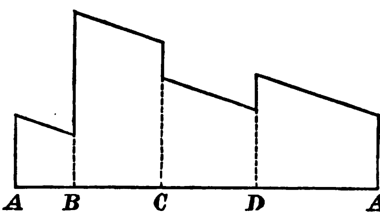


18. If all the changes of potential that occur throughout a given battery at all the junctions of metals and liquids or dissimilar metals or dissimilar liquids are known, we may represent them on the diagram. Suppose for instance we have a closed circuit containing a single Volta's cell composed of an amalgamated zinc plate and a copper plate in dilute sulphuric acid, the poles of the battery being joined by a copper wire. Now the real variations of potential between surfaces in contact in such a circuit are not well known. Measures have been made by electrostatic methods, but it has been shown that the gas in which the substances are contained has great effect on the result obtained; in fact zinc, with regard to copper, appears to have in air a positive potential, and in hydrogen sulphide a negative potential. It is very probable that what is measured in this case is the P. D. between layers of some chemical compounds formed by the action of the gas in which they are, on the surfaces of the two substances in contact.

We shall see later on what conclusions may be drawn from thermoelectricity, with regard to the P.D. of metals in contact.

As an illustration of the use of the diagram, let us take as the P.D.s between the substances in contact in the Volta's cell those that are obtained by electrostatic methods by making the measures in air.

At the junction of the copper wire and zinc plate, there is a sudden rise of potential, from the copper to the zinc. The potential suddenly falls in passing from the zinc plate to the liquid, and rises by about an equal amount in passing from the liquid to the copper plate. So that the variations of potential will be represented as in the figure, A and A' denoting some point of the copper wire, B the junction of the copper wire and zinc plate, C that of the zinc plate and the liquid, and D that of the liquid and the copper plate.



19. Measure of very great Resistance by means of Electrometer and Condenser. Let R be the value of the resistance to be measured. Let the terminals of the given resistance be connected to the conductors of a charged condenser, of capacity C, and let these at the same time be connected to the electrodes of an electrometer. Let Q be the charge of the condenser, and E the P.D. of its surfaces. Then if I is the current flowing through the given resistance,

$$Q = CE,$$

$$E = RI = -R\frac{dQ}{dt}.$$

Thus
$$Q = -CR\frac{dQ}{dt},$$

$$\therefore\ Q = Q_0 e^{-\frac{t}{C}},$$

where Q_0 is the charge when $t = 0$.

$$\therefore E = E_0 e^{-\frac{t}{CR}},$$

where E_0 is the initial P. D. of the condenser surfaces.

Or taking the electrometer readings θ_0, θ to be proportional to the P. D.s, we get

$$\log\theta_0 - \log\theta = \frac{t}{CR},$$

$$R = \frac{t}{C(\log\theta_0 - \log\theta)}.$$

As the condenser itself cannot be considered a perfect insulator, two experiments must be made; the first to determine the condenser resistance R_1, the second to determine the entire resistance R_2 when the surfaces are connected to the ends of the given conductor. Then the resistance R of the conductor is given by

$$\frac{1}{R} = \frac{1}{R_2} - \frac{1}{R_1}.$$

CHAPTER V.

ELECTROLYSIS.

1. IT is found that whenever an electric current is passed through a liquid which is a chemical compound, the liquid is decomposed. This decomposition is called *electrolysis*, and the liquid which is decomposed is called an *electrolyte*. Let us consider, as an example, the electrolytic decomposition of water. To make water a conductor we must mix with it a little sulphuric acid. Let the current then be led into and out of the liquid by two platinum plates, which are called *electrodes*. The water will be decomposed into its component gases, hydrogen and oxygen; and if two closed inverted tubes, previously filled with water, are placed over the electrodes, the gases may be collected in them. The hydrogen will come off from the electrode by which the current leaves the liquid, which is called the *kathode*, and the oxygen from the electrode by which the current enters the liquid, which is called the *anode*. It is found that the quantity of water that is decomposed, or the quantity of hydrogen or oxygen that is given off, is exactly proportional to the quantity of electricity that passes, that is, if the current is uniform, to the product of the strength of the current and the time for which it passes.

2. **Hypothesis of Grothüss.** This liberation of the components of the water is explained by the following hypothesis. The elements in the molecule of water (H_2O), and in that of any other compound liquid, are supposed not to be inseparably united to each other, but to be constantly becoming dissociated

and joining up with other atoms of oxygen and hydrogen. When no current passes the recombinations just make up for the dissociations, and the state remains unchanged; but the electric current gives a definite direction to the movements of the atoms, and they come off uncombined at the electrodes.

In the same way if we decompose any compound liquid, or solution of a compound, by electrolysis, the component parts, called *ions*, will go to the electrodes, and, except when secondary chemical actions occur, may be collected there. The quantity of the compound that is decomposed is always exactly proportional to the quantity of electricity that has passed. The metal or base of the compound always goes with the current, and appears at the *kathode* of the electrolytic cell or vessel in which the decomposition is taking place.

3. Faraday, who has carefully investigated the conditions of electrolysis, has given the two following Laws.

LAW I. The quantity of an electrolyte decomposed per second by a current is directly proportional to the current strength.

This law suggests that the way in which electricity passes through an electrolyte is this: that each atom of the component part of the electrolyte that goes towards the kathode, carries with it a definite quantity of positive electricity, and each atom that goes towards the anode, carries with it a definite quantity of negative electricity. It also shows that electricity can pass through an electrolyte in one way only, namely by electrolysis, and that an electrolyte never conducts electricity in the same way as an ordinary metallic conductor. There is a marked difference in the effect of change of temperature on the resistance in the two cases, a rise of temperature increasing the resistance of a metallic conductor, but diminishing that of an electrolyte.

The second law relates to the proportions in which different electrolytes are decomposed by the same quantities of electricity. It is this;

LAW II. If the same quantity of electricity is passed through several electrolytes, the quantities by weight of the various components liberated are proportional to their chemical equivalents.

The weight of an atom is supposed to be proportional to the chemical equivalent (or *a* chemical equivalent, if the substance has different valencies) of the substance. Thus the number of atoms of each ion will be the same, if the valencies are the same, that is if the same number of atoms in each ion can replace each other in combinations. This shows that for this case a certain definite quantity of electricity liberates, or passes with, an atom of each ion, whatever the ion may be.

Now suppose we have two electrolytes in which there are two bases of different ions of different valencies. Suppose we have a silver and a copper salt, the silver being monovalent, and the copper divalent, that is forming a cupric salt. The quantities by weight of silver and copper liberated by the same quantity of electricity are proportional to their chemical equivalents, that is to their atomic weights divided by their valencies, or are in the ratio $108 : 63 \div 2$, or $108 \times 2 : 63$. But the atoms having weights in the ratio $108 : 63$, we must suppose that the same quantity of electricity which liberates two atoms of silver, liberates one atom of copper.

DEF.: **THE ELECTRO-CHEMICAL EQUIVALENT of a substance is the mass of it in grammes deposited per second by an ampere.**

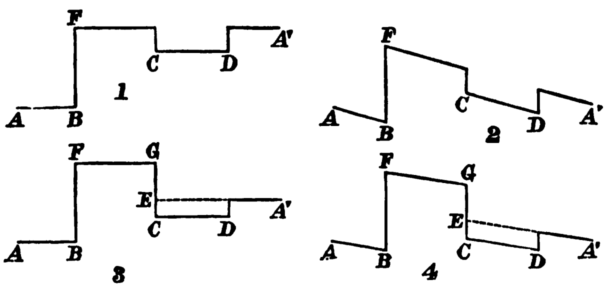
Thus electro-chemical equivalents are proportional to chemical equivalents.

4. Back E. M. F. of polarization. It is found that if a current is passed through an electrolytic cell such as that in which water is decomposed by means of platinum electrodes, the current starts at a maximum value which immediately falls off. This is found to be due, not so much to an increase of resistance in the electrolytic cell, as to an E. M. F. generated in it which tends to send a current in the opposite direction. If the battery is taken off and the wires from the cell joined up, a

current will be found to pass in the opposite direction. This E. M. F. generated in the electrolytic cell is called the back E. M. F. of polarization, and for every cell there is a certain maximum value to which it can rise. It may happen that this maximum value is greater than that of the battery used for the decomposition. Suppose, for instance, we attempt to decompose water with a single Daniell cell. A current passes for a very short time, but the electrodes soon become covered with the gases, and are then said to be polarized. The back E. M. F. has become equal to the E. M. F. of the Daniell cell and the current stops. But if two Daniell cells in series are used, the E. M. F. of polarization can never become equal to the E. M. F. of the two, and the decomposition will go on.

As soon as the current begins to pass, a fall of potential from the anode to the kathode of the battery takes place, in addition to that due to the passage of the current through the resistance of the cell; and if this fall becomes equal to the E. M. F. of the battery, the current stops.

The variations of potential throughout the circuit may be represented graphically as in the figures.



Figures 1 and 2 represent the potentials before polarization, and figures 3 and 4 after polarization. Figures 1 and 3 refer to open circuit, and figures 2 and 4 to closed circuit. *A* and *A′* represent points which are joined on closed circuit, and therefore in figures 2 and 4 they are on the same horizontal level. The rise of potential due to the E. M. F. of the battery is repre-

sented as a single variation at B, BF denoting this E. M. F. The variations of potential between the electrodes and the liquid are represented at C and D. The E. M. F. of polarization is denoted by the line EG in figures 3 and 4. The falling off of the current is shown by the difference of the slopes of the sloping lines in figures 2 and 4, the slope in 4 being less than in 2.

If we perform a decomposition, such as that of copper sulphate with pure copper plates, so that on the whole the chemical composition of the substances in the cell is not altered, then there is no E. M. F. of polarization. In this case copper is deposited on the kathode, and an equal quantity of copper is dissolved off the anode, keeping the constitution of the liquid constant.

5. Let us consider a portion of a circuit of resistance R, including an electrolytic cell of E. M. F. E, and in which a current I is passing. The entire fall of potential along this portion of the circuit is $RI+E$. Therefore the energy expended per second is RI^2+EI. The portion RI^2 is spent in heating the circuit, and may be considered as lost; the portion EI is spent in producing chemical decomposition, and we shall see that it may be obtained again in the form of electrical energy. By diminishing the value of I we may make the term RI^2 as small as we please in comparison with EI, so that we may get practically the whole of the energy expended in producing decomposition and none of it wasted in heat. The energy expended in producing the decomposition is stored up in the uncombined substances as chemical potential energy, to be given out by them again, in some form, when they recombine. And the energy spent in separating the substances and that which they produce on recombining are always exactly equivalent.

6. Suppose Q coulombs of electricity to pass through the cell, dissociating an ion of electro-chemical equivalent z. The mass of the ion dissociated will be zQ grammes. Let H be the heat of combination, measured in calories, of one gramme of the ion

with that from which it has been dissociated: J the mechanical
4.19 equivalent of heat, ~~·4~~. Thus the heat of combination of zQ grammes, measured in joules, is $zQJH$. And if E is the E. M. F. of the cell, the work done, in joules, to produce the dissociation is QE.

$$\text{Thus} \quad QE = zQJH;$$
$$\therefore \; E = zJH.$$

This enables us to calculate the E. M. F. of polarization in any electrolytic decomposition.

It is from the same source, namely, chemical combination, that the energy of any electrical battery is derived; and if we know accurately the heats of combination of all the substances in the cell we can calculate its E. M. F.

Suppose we have a circuit of resistance R in which is any number of battery cells and voltameters, or electrolytic cells, that is vessels in which electrolysis is going on, and let the electro-chemical equivalents of the substances going into combination be x_1, x_2, &c.; let their heats of combination be H_1, H_2, &c. Let the electro-chemical-equivalents of the substances coming out of combination be y_1, y_2, &c.; their heats of combination h_1, h_2, &c. Let I be the current passing. Then the energy expended per second is $JH_1x_1I + JH_2x_2I + \ldots$. And the work done in producing chemical decompositions is $Jh_1y_1I + Jh_2y_2I + \ldots + RI^2$. Thus we have the relation

$$JI\Sigma(xH) = JI\Sigma(yh) + RI^2.$$
$$\text{Or} \quad J\Sigma(xH) - J\Sigma(yh) = RI.$$

This gives for the E. M. F. in such a circuit the value

$$J\Sigma(xH) - J\Sigma(yh).$$

We may denote the E. M. F. of a cell or combination of cells by the formula $J\Sigma(zH)$, z being taken positive or negative according as it refers to a substance going into or coming out of combination. If we measure H in joules and not calories, the formula is $E = \Sigma(zH)$.

This theory was given by Sir William Thomson (now Lord Kelvin).

7. This formula in many cases gives a result considerably different from the measured E. M. F. of the cell, and so the following theory has been developed by Helmholtz and Lippmann.

Suppose we have a cell that can be reformed by reversing the current. Then by making the current, I, indefinitely small, the energy expended per second by the cell in the form of heat (RI^2) disappears in comparison with the entire energy expended, which is proportional to I. Thus the cell may be made to perform a Carnot's cycle.

Now the condition of the cell may be fixed by three independent variables:

(1) Temperature (thermo-dynamic) T;

(2) Quantity of electricity that has passed through it x;

(3) Degree of concentration of its solutions.

This may be reckoned by supposing the cell contained in a cylinder with a piston, the cylinder being always full of aqueous vapour at maximum pressure. Let the volume of cell and enclosure which will thus determine the degree of concentration be v.

Now let dH (in joules) be an infinitesimal increment of heat given to the cell, and dW an infinitesimal quantity of work done by it. Then if p denotes pressure, c thermal capacity of cell, E its E. M. F., and l_1, l_2 are other co-efficients, we have

$$dW = p\,dv + E\,dx,$$
$$dH = c\,dT + l_1\,dx + l_2\,dv.$$

Let us suppose v to remain constant, so that we shall not consider the effect of variations of concentration. Then for the increment of intrinsic energy, and the increment of entropy in the cell, we have the expressions,

$$\begin{aligned} dQ &= dH - dW, \\ &= c\,dT + (l_1 - E)\,dx, \end{aligned} \qquad (1)$$

$$\frac{dH}{T} = \frac{c}{T}dT + \frac{l_1}{T}dx. \qquad (2)$$

By the first and second Laws of Thermodynamics each of these expressions is a complete differential,

$$\therefore \quad \frac{dc}{dx} = \frac{d}{dT}(l_1 - E),$$

$$\text{and} \quad \frac{d}{dx}\frac{c}{T} = \frac{d}{dT}\frac{l_1}{T},$$

$$\text{or} \quad \frac{dc}{dx} = \frac{dl_1}{dT} - \frac{l_1}{T}.$$

$$\therefore \quad l_1 = T\frac{dE}{dT}.$$

$$\text{But} \quad l_1 = \left(\frac{dH}{dx}\right)_T.$$

Thus to keep the cell at constant temperature we must supply heat $T\frac{dE}{dT}$ to it for each unit of electricity that passes through it: or the work done by the cell exceeds the energy of the chemical combinations by this amount.

$$\therefore \quad E = \Sigma\,(zH) + T\frac{dE}{dT}.$$

This shows that Thomson's law would only be true for cells in which the E. M. F. does not vary with the temperature. In this case from an equation above we see that $\frac{dc}{dx} = 0$. That is, the thermal capacity of the cell remains constant whether its constituents are combined or uncombined, that is they obey the law of Woestyn.

This theory agrees with experiment in showing whether E or $\Sigma\,(zH)$ is the greater; but the quantity $T\frac{dE}{dT}$ is always numerically less than the measured difference between E and $\Sigma\,(zH)$.

8. If the voltameters are such as produce a back E. M. F. by polarization, and we take them before any decomposition has taken place, and if the normal value of $\Sigma\,(yh)$ is greater than $\Sigma\,(xH)$, then $\Sigma\,(yh)$ will have a very small value at first, but

will rapidly increase, and there will be a current to start with, which will soon disappear.

Let us consider as an instance of the gradual rise of E. M. F. of polarization the decomposition of water with platinum electrodes. We know that the E. M. F. is due to the energy of the chemical recombination of the gases, which energy we get stored up in the cell on account of the gases being in a state of separation. Now when the current has only just begun to pass, the electrodes having been put in clean and free from gas, the E. M. F. of polarization being smaller than its normal value, we must infer that the work done in separating unit mass of hydrogen at first from its combination with oxygen is less than that required after decomposition has gone on for some time. And in fact this is the case, for the gases do not at first come off so as to be in a complete state of separation, but condense on the electrodes, forming with them what are very analogous to chemical compounds. So that we would not at first get all the energy of recombination as the gases are already in a state of partial chemical combination. That is, we do not get the full E. M. F. of polarization.

It may happen that, using a single Daniell cell to decompose water, a permanent current, but an extremely feeble one, is obtained. This is due to the gases on the electrodes being constantly diffused into the liquid, and just sufficient current passes to keep the electrodes in their state of partial polarization.

9. Secondary Actions. It often happens that the direct products of an electrolytic decomposition are not obtained at the electrodes, but that they act on other substances in presence of them and so give rise to other products at the electrodes. These actions are called *Secondary Actions.* If we decompose copper sulphate with platinum electrodes $Cu\,SO_4$ is broken up into Cu and SO_4. Copper is deposited on the kathode, and sulphion, SO_4, acting on water, gives off oxygen at the anode and forms sulphuric acid, which is found in the neighbourhood of the anode.

10. Voltameter. A very convenient way of measuring the strength of a current is by means of the electrolytic liberation by it of some substance whose electro-chemical equivalent is known, and under circumstances in which secondary actions are avoided. An electrolytic cell used for this purpose is called a *voltameter*. Voltameters in which the substance liberated is hydrogen or copper are frequently used, but Lord Rayleigh has found that the voltameter giving the most accurate results is that in which silver is deposited from a 15 per cent. solution of silver nitrate, the mass of silver deposited in one second by an ampere being ·001118 gramme.

CHAPTER VI.

Thermo-Electricity.

1. Seebeck Effect. It was discovered by Seebeck that, in a circuit composed of two different metals, if the two junctions be kept at different temperatures a permanent current is maintained in the circuit.

2. Peltier Effect. The converse of this phenomenon was discovered by Peltier. If a current be passed round a circuit composed of two different metals there will be heating at one junction and cooling at the other.

3. Let us consider these two effects in the same circuit. If a heating at junction A and a cooling at junction B produce a current in a certain direction, then if a current be passed in the same direction by means of a battery placed in the circuit, cooling will be produced at A and heating at B. So that a current passing in a compound circuit according to the Seebeck effect cools the hot junction and warms the cold one, and thus tends to stop itself.

The metals showing these phenomena best are antimony and bismuth. If in an antimony-bismuth circuit, at ordinary temperatures, a current is passed, the junction at which the current passes from antimony to bismuth is heated and the other cooled. And conversely, if the junctions are kept at different temperatures a current will pass from bismuth to antimony across the hotter junction.

In discussing these thermo-electric currents one must consider how the potential varies from point to point in a compound circuit. When two metals are placed in contact there is

as a rule a sudden variation of potential in passing from one to the other across the junction, and this variation depends upon the temperature of the junction as well as upon the two metals. Again, if the various parts of a single metal are kept at various temperatures they will also, as a rule, be at various potentials. Thermo-electric currents may be due to either or both of these causes of variation of potential.

4. The following laws with regard to compound circuits have been proved by experiment.

I. LAW OF VOLTA. In a compound circuit, consisting of any number of different metals, all points of which are at the same temperature, there is no current.

This law shows that if there are variations of potential in passing from one metal to another these are always exactly compensated in passing back to the metal from which we started, whatever different metals we pass through, and in whatever order, provided we keep all the junctions at the same temperature.

II. LAW OF MAGNUS. In a homogeneous circuit, however the temperature varies from point to point, there is no current.

This law shows that if the potential varies from point to point with the temperature, it always returns to the same value on passing along the circuit so as to come to a point at the same temperature as that from which we started. Thus, given the potential at a point of given temperature, the potential of any other point depends merely on its temperature and not on the manner in which the temperature has varied from the starting-point to it.

From this law it follows that the E. M. F. in a compound circuit depends merely on the temperatures at the junctions, for the E. M. F. between the extremities of a homogeneous conductor depends only on the temperatures of the extremities.

5. Suppose we have a circuit made of two metals A and B, the junctions being kept at temperatures t_1 and t_2. Let us denote the E. M. F. tending to drive a current from A to B across

the junction at temperature t_2 by the symbol $E_{t_1}^{t_2}(AB)$, and use similar symbols for the E. M. F.s of other compound circuits formed of two metals. Then we have the following relations, as results of experiment.

If θ is any temperature intermediate between t_1 and t_2,

$$E_{t_1}^{t_2}(AB) = E_{t_1}^{\theta}(AB) + E_{\theta}^{t_2}(AB).$$

This is called the **LAW OF SUCCESSIVE TEMPERATURES.**

The symbol $E_{\theta}^{t_2}(AB)$, θ being considered constant, is some function of t_2, and $E_{t_1}^{\theta}(AB)$ is plainly the same function of t_1 with the opposite sign. Thus $E_{t_1}^{t_2}(AB)$ can be expressed in the form

$$E_{t_1}^{t_2}(AB) = F(t_2) - F(t_1).$$

If C is any third metal,

$$E_{t_1}^{t_2}(AB) = E_{t_1}^{t_2}(AC) + E_{t_1}^{t_2}(CB).$$

This is called the **LAW OF INTERMEDIATE METALS.**

This may be proved by showing that the E. M. F. of a circuit of two metals A and B with junctions at temperatures t_1 and t_2 is not altered by introducing a conductor of the metal C into either of the junctions and keeping it all at the temperature of the junction. Then we may suppose a point in the conductor C in the junction at temperature t_2 to be carried to temperature t_1. By the law of Magnus this makes no difference in the E. M. F. of the circuit, and we have now the two E. M. F.s, $E_{t_1}^{t_2}(AC)$ and $E_{t_1}^{t_2}(CB)$, in the circuit.

We see from this that a compound circuit will have the same E. M. F. whether its various parts are connected directly or soldered together, the introduction of the solder into a junction making no difference.

These two laws are due to Becquerel.

6. Phenomenon of Inversion. If we have a circuit formed of two metals, and keep one junction at a constant temperature

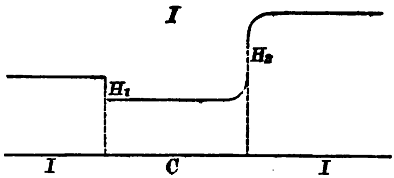
and gradually raise the temperature of the other, the E. M. F. of the circuit will, as a rule, rise to a maximum value, then diminish to zero, and then become negative. The temperature at which the E. M. F. changes sign is called the temperature of inversion corresponding to the given temperature of the colder junction.

It can be shown that differences of potential at the junctions alone are not sufficient to account for thermo-electric currents. For suppose in a circuit of two metals there is a rise of potential H_2 at the hotter junction and a fall H_1 at the colder, reckoned in the direction in which the current goes, so that H_2 must be greater than H_1. At first the current tends to cool the hot junction and heat the cool one; but by gradually increasing the temperature of the hot junction we pass the temperature of inversion, and a current passes in the opposite direction cooling the cold junction and heating the hot one. So that if we were sufficiently to get rid of loss of heat by radiation the current would keep itself up and tend to increase itself, transporting heat from the cold to the hot junction. This would be a violation of Carnot's principle. Thus the hypothesis of variations of potential at the junctions alone is untenable.

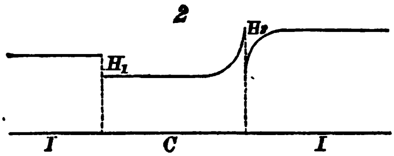
7. We must suppose then, with Sir Wm. Thomson, that there are other variations of potential in the circuit, namely, variations along a homogeneous conductor whose parts are at different temperatures. Suppose we have a copper-iron circuit with its two junctions at temperatures t_1 and t_2, t_2 being the greater, and both being at first below the temperature of inversion. Let us suppose that all points of the circuit are at temperature t_1, except those in the vicinity of the hotter junction. Let us represent in a figure the changes of potential on open circuit, that is, when a division is made somewhere in the iron. Let there be a sudden drop of potential H_1 from iron to copper at the cold junction, and a sudden rise H_2 at the hot one. There are, besides, variations of potential in both the copper and iron in the vicinity of the hot junction, which must be represented, as we shall see, as the figure is drawn, that is, there is a rise of

potential in the copper but a fall in the iron as we pass to points of higher temperature.

Let us denote the rise in potential in copper and iron together, due to the variation of temperature, by h. Then the entire rise of potential at and in the vicinity of the hot junction is H_2+h. And the E.M.F. of the circuit is H_2+h-H_1.



Now as we raise the temperature of the hot junction H_2+h increases to a maximum value, then diminishes, and when t_2 has passed through the value of the temperature of inversion corresponding to t_1, H_2+h-H_1 becomes negative, that is the current changes in direction. We shall see that h is always positive, and it is only H_2 that becomes negative. We must in these cases represent the changes of potential as in the second figure.



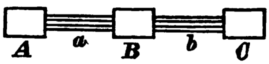
8. When a current is passing across a junction at which there is a fall of potential H, for every unit of electricity (coulomb) that passes the junction electrical energy H is expended, and as this appears merely in the form of heat at the junction, a quantity of heat H, measured in joules, is developed. This is, of course, in addition to the heat spent in overcoming the resistance of the circuit, and which is developed throughout the circuit. If now the direction of the current be reversed, for every unit of electricity that crosses the junction, rising in potential by an amount H, electrical energy H is developed, and thus a quantity of heat H is lost at the junction.

Now to consider the variations of potential in the other parts of the circuit, let the rise of potential in a given conductor in passing from a point at temperature t to one at an inde-

finitely near temperature $t + dt$ be σdt, where σ is a quantity depending on the nature of the conductor, and which may also be a function of t. For a unit of electricity passing from the point at temperature t to that at temperature $t + dt$ heat σdt is absorbed; and if it passes in the opposite direction, heat σdt is developed, besides that due to the resistance of the circuit.

Thus in an unequally heated metal in which σ is positive, the passage of an electric current from places of low to places of high temperature should cause a cooling, and from places of high to places of low temperature, a heating. If σ is negative the opposite effects should be observed. And these effects are in addition to the ordinary heating effect in accordance with Joule's Law.

9. Thomson Effect. Sir Wm. Thomson has made direct experiments to determine whether any such phenomena exist as those we have just alluded to. A bundle of iron wires was passed through two vessels of boiling water A, C, and a vessel of cold water B between A and C. The temperatures at a and b were taken with the current passing in either direction along the wire. The temperature at a was observed to be greater when the current passed from C to A than when it passed from A to C; and the temperature at B was less when the current passed from C to A. It follows that in iron σ is negative: in copper it is positive.



Experiments on several metals have been made by Le Roux, who has found that the heating effect is proportional to the current. He has also found that in lead alone of all metals the value of σ is practically zero.

The quantity σ has been called by Thomson *the specific heat of electricity* for the given metal.

10. Sir Wm. Thomson has expressed the E.M.F. in a thermo-electric circuit by means of a function which he calls *the thermo-electric power* of the two metals.

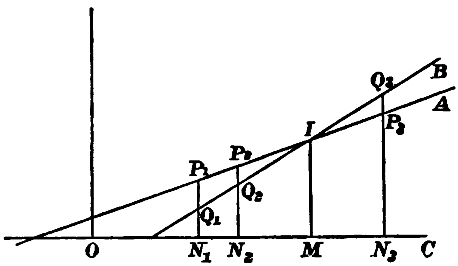
DEF. If a circuit is formed of two metals with the junctions at indefinitely near temperatures, t and $t + dt$, and

dE is the E. M. F. of the circuit, then the differential coefficient $\frac{dE}{dt}$ is called the THERMO-ELECTRIC POWER of the two metals for the temperature t.

Let us suppose two loci A, B to be drawn to coordinate the thermo-electric powers of two metals A, B with respect to a third one C, which we will suppose to be lead, with the temperatures; the temperatures being taken as abscissae, and the thermo-electric powers are ordinates. Then for any given temperature, the thermo-electric power of A and B can be found by taking the intercept of the ordinate between these two lines. For by the law of intermediate metals, we have

$$E(AB) = E(AC) - E(BC).$$

$$\therefore \quad \frac{dE(AB)}{dt} = \frac{dE(AC)}{dt} - \frac{dE(BC)}{dt}.$$



Thus at the temperature denoted by ON_1, N_1P_1 and N_1Q_1 being the thermo-electric powers of A and B with respect to C, that of A with respect to B is denoted by P_1Q_1.

A is said to be thermo-electrically positive with respect to B at the temperature t, when one junction being at t and the other at $t+dt$, the current passes from A to B across the hotter junction. The line A will fall above B in the diagram, for temperatures for which A is thermo-electrically positive with respect to B.

11. Suppose we have a circuit formed of A and B with the junctions at temperatures t_1 and t_2, denoted by ON_1 and ON_2. Let us denote by ϕ (t) the thermo-electric power at temperature t.

Then by the law of successive temperatures, if E is the E. M. F. of the circuit,

t.

$$E = \int_{t_1}^{t_2} \phi(t)\, dt.$$

Thus E is represented by the area $P_1P_2Q_2Q_1$.

If the hot junction is at a temperature denoted by ON_3 the E. M. F. will be denoted by $IQ_1P_1 - IQ_3P_3$. Thus drawing the ordinate IM from the point I where the lines A and B intersect the temperature of the hot junction giving the maximum E.M.F. is that denoted by OM.

It is found by experiment that the E.M.F. of a given circuit of two metals, t_o and t being the temperatures of the two junctions, and t_n some other temperature depending on the two metals, is given by the formula

$$E = k(t - t_o)\left(t_n - \frac{t_o + t}{2}\right).$$

From this $$\frac{dE}{dt} = k(t_n - t).$$

Thus the maximum value of the E. M. F. is obtained by making $t = t_n$; which shows that the temperature t_n is that denoted by OM.

Making t_o and t indefinitely near we see that the thermo-electric power at t is

$$\phi(t) = k(t_n - t),$$

which shows that the lines drawn to represent the thermo-electric powers at various temperatures are straight lines.

The thermo-electric power for temperature t_n is zero. This temperature, t_n, is called the *neutral point* for the two metals.

It appears from the expression for E that the temperature of the cold junction and the temperature of inversion are equally distant from the neutral point.

12. Suppose we have a circuit formed of two metals A and B, with the junctions at temperatures t_1 and t_2, all temperatures being measured on the thermodynamic scale. Let H_1, H_2 be the falls of potential in passing from A to B at these two junctions

respectively. Let σ, σ' be the specific heats of electricity in A and B. Then the quantity of heat absorbed in the circuit for the passage of unit quantity of electricity from B to A across the junction at temperature t_2, neglecting that due to the resistance, is

$$H_2 - H_1 + \int_{t_1}^{t_2} (\sigma' - \sigma)\, dt,$$

which is equal to the E. M. F. E of the circuit.

Now by indefinitely diminishing the current, the heat developed in accordance with Joule's law becomes indefinitely small as compared with that developed or absorbed on account of thermo-electric variations of potential. Thus when the current is reversed there will be absorption of a quantity of heat equal to that developed in this case for each unit of electricity that passes. Thus the circuit acts like a perfectly reversible engine when the temperatures of the junctions are indefinitely near together. Therefore the algebraical sum of the quantity of heat absorbed at each point divided by the temperature of that point is zero.

Thus $$\frac{H_2}{t_2} - \frac{H_1}{t_1} + \int_{t_1}^{t_2} \frac{\sigma' - \sigma}{t}\, dt = 0.$$

Or since t_1 and t_2 are indefinitely near together,

$$\frac{d}{dt} \cdot \frac{H}{t} + \frac{\sigma' - \sigma}{t} = 0.$$

$$\therefore \quad \frac{H}{t} = \frac{dH}{dt} + \sigma' - \sigma.$$

And the thermo-electric power $\phi(t)$ at temperature t is

$$\frac{dE}{dt} = \frac{dH}{dt} + \sigma' - \sigma.$$

$$\therefore \quad \phi(t) = \frac{H}{t} \cdot$$

It follows from this, that if the junction of two metals is at their neutral point, at which their thermo-electric power is zero, there is no sudden variation of potential at the junction, and

no absorption or development of heat there. Thus in the case we have considered of a copper-iron circuit, when $H_2 + h$ has its maximum value, H_2 is zero.

13. Suppose we have a circuit of two metals with both junctions at first below the neutral point. The passage of the current cools the hot junction and warms the cold one. Keeping the lower temperature constant and gradually raising the upper one, we have just seen that no thermal effect takes place at the hot junction when it is at the neutral point. If we further increase its temperature, the passage of the current now heats both junctions; but when the temperature of inversion is passed the direction of the current is changed and it now cools both junctions. If we now raise the temperature of the cold junction above the neutral point, the passage of the current again cools the hot and warms the cold junction.

CHAPTER VII.

Electro-Magnetic Induction.

1. If we have an electric circuit carrying a current i in a magnetic field, and such that the quantity of magnetic induction passing through it in the positive direction due to the field is N, we have seen that the mutual potential of the field and circuit is $-iN$.

Suppose the circuit to move from a position in which the quantity of induction enclosed by it is N_1 to one in which the quantity is N_2: the work done on it by the field is the diminution of potential, that is, it is $i(N_2-N_1)$. Now the field always tends to move the circuit so as to do positive work upon it, so that it tends to move it so as to make it enclose the largest possible quantity of magnetic induction in the positive direction.

2. Now suppose that the magnetic field in which the circuit is is due to another electric circuit: let us then have two electric circuits carrying currents i_1 and i_2. Let the quantity of induction through the first due to the second be N_1, and that through the second due to the first N_2. Then the mutual potential of the two circuits is equal to $-i_1N_1$ or to $-i_2N_2$. Thus these expressions are equal; and so we see that when the currents are equal the quantity of induction through each circuit due to the other is the same. Let us denote the quantity of induction through either circuit due to unit current in the other by the symbol M. Now the magnetic intensity at any point due to a given current in the second circuit, and therefore also the quantity of induction through the first circuit due to a given

current in the second, is generally proportional to the current in the second, and therefore when this current is i_2 the quantity of induction through the first circuit due to it is Mi_2. Thus the mutual potential of the two circuits is $-Mi_1 i_2$.

The quantity M is called the *coefficient of mutual induction* of the two circuits; and we have for its definition:

DEF.: **THE COEFFICIENT OF MUTUAL INDUCTION of two circuits is the quantity of magnetic induction passing through either of them per unit current in the other.**

M is also called the MUTUAL INDUCTANCE of the two circuits.

M is also the work which must be done on either circuit, against the action of unit current in each, to take it away from its given position to an infinite distance from the other.

Or it is the work which would be done by either circuit on the other, in consequence of unit current in each, as the other moves from an infinite distance off up to its given position relatively to the first.

3. M will depend on the form, size, and relative position of the two circuits; and on the magnetic susceptibilities of neighbouring substances. And it should be noticed that M, or the quantity of induction through either circuit per unit current in the other, will not be constant if we have substances in the field of variable magnetic susceptibility, as iron or steel. In such a case the value of M will vary with the currents used; and it can only be regarded as constant for such currents as will produce very small magnetic intensity in the iron or steel.

4. Suppose we know the value of M for two circuits having one turn each; and now we give to the circuits n_1 and n_2 turns, all the turns occupying practically the same position as the single turn of each circuit occupied before. This will produce the same effect as if we multiplied the currents in the two circuits by n_1 and n_2. Thus the coefficient of mutual induction is $n_1 n_2$ times what it is if each circuit has only one turn.

5. Solenoid. A single circular circuit of area s and carrying a current i has a magnetic moment is. m such circuits placed at equal distances along a common axis have magnetic

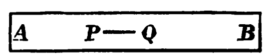
moment msi. Such an arrangement is an ideal solenoid. But as it cannot be realized in practice, a practical solenoid is a spiral with its ends brought back so as to be parallel to its axis. Suppose we form such a spiral with m turns all lying on the surface of a cylinder of cross-section s, and let i be the current carried. The magnetic moment of the arrangement is $m s i$, being practically equivalent to that of an ideal solenoid. Each circular circuit acts like a shell of area s and strength i, and the solenoid may be considered as equivalent to a cylindrical magnet of the same size and position built of magnetic shells, or discs, not necessarily very thin. If there are n turns per unit length, the thickness of each disc would be $t = 1/n$. And if σ is the surface density of each shell $i = \sigma t = \sigma/n$, $\therefore \sigma = i n$. If the magnetic cylinder is built up of these discs wherever the surfaces of two cylinders with magnetic densities σ and $-\sigma$ are in contact, these two densities neutralize each other and the solenoid is equivalent to a cylinder of the same size and position having surface densities on its ends equal to $i n$ and $-i n$.

This cylindrical magnet may be considered as equivalent to the solenoid with regard to its action at all points outside it.

Let us consider the magnetic intensity at a point within the solenoid.

First, let us suppose the solenoid to be of indefinite length. Let AB be the positive direction of the solenoid, and suppose a unit magnet pole to start from a point P at a practically infinite distance from either end, and move through unit length from P to Q parallel to the axis of the solenoid, and in the positive direction of the lines of force.

The work done on the unit pole by the solenoid is the same as that which would be done by the pole on the solenoid if this moved through unit length in the direction



BA, or the same as would be done by the pole on unit length of the solenoid which moved from B to A. Now in this motion the solid angle subtended at the pole by the positive face of each turn of the solenoid which has moved has diminished from

4π to o. Thus the work done on each turn is $4\pi i$. And the work done on the n turns in the unit length by the pole is $4\pi ni$. This is also the work done on the pole by the solenoid in case the pole moves through unit length. And if the pole moved from P in any other direction than parallel to the axis it would have to move a greater distance for the performance of the same amount of work. Thus the magnetic intensity at any point P inside the solenoid and at a distance from its ends practically infinite compared with its diameter is $4\pi ni$.

6. Next, suppose that the solenoid AB cannot be considered of infinite length. If we continued it by means of two infinite solenoids added on at A and B of the same cross section, number of turns per unit length, and current as itself, the magnetic intensity at P would be $4\pi ni$. But these would be equivalent in their action at P to two infinite cylindrical uniform magnets of the same cross section as the given solenoid and of magnetic moment ni per unit length. Thus their action at P is the same as that of two layers of magnetism at A and B coinciding with the ends of the solenoid, and of densities ni and $-ni$ respectively. Thus in this case the magnetic intensity at P is $4\pi ni$ — the intensity at P due to these two layers at the ends of the solenoid.

7. Let the length of the solenoid be l, so that $m = ln$. Suppose we place within a concentric cylindrical core of length l, the ends of the core and the coil being coincident. Let the magnetic susceptibility of the core be κ; its cross-section s'. Let the length l be such that we may take the magnetic intensity throughout the space inside the coil to be $4\pi ni$. The induced magnetization of the core is $4\pi ni\kappa$. So that the core becomes a magnet of moment $4\pi ni\kappa ls'$. And the solenoid with its core has magnetic moment

$$\begin{aligned} &nils + 4\pi ni\kappa ls' \\ &= mi(s + 4\pi\kappa s'). \end{aligned}$$

Thus by the addition of the core the magnetic moment is changed in the ratio $s : s + 4\pi\kappa s'$.

Suppose a coil of n' turns wound on upon the solenoid at

a great distance from its ends. The induction through a turn of this coil when there is no core is $4\pi nsi$; and when there is a core it is $4\pi n(s+4\pi\kappa s')i$. Thus the coefficients of mutual induction of the two coils in the two cases are

$$4\pi nn's, \text{ and } 4\pi nn'(s+4\pi\kappa s').$$

8. Induced Currents. Faraday discovered that while a closed electric circuit is moving relatively to a magnetic system a current is in general set up in the circuit. This he called an *induced current.*

With regard to the direction of induced currents, the following law was given by Lenz:

When a conductor is moving in a magnetic field a current is induced in the conductor in such a direction as by its mechanical action to oppose the motion.

It appears from Faraday's experiments that any increase or diminution of the quantity of induction through a closed circuit will produce an induced current in it. Thus a current is produced in a coil, as a rule, by increasing or decreasing the current in a neighbouring coil. If we take a neighbouring coil and suddenly start a current in it its effect in inducing a current in the given coil is the same as if we suddenly brought it up to its position from an infinite distance off with its current in it. If we suddenly stop the current the effect is the same as if we suddenly carried the coil off to an infinite distance with its current in it.

9. Suppose we have a closed electric circuit of resistance R, containing an E. M. F. E, and carrying a current I; and let the quantity of induction through it in the positive direction at any instant be N. Now if N is constant, we have the ordinary relation $E=RI$. But if N varies let us suppose that its rate of increase is $\frac{dN}{dt}$. Then $I\frac{dN}{dt}$ is the rate of decrease of the potential of the circuit with regard to the field in which it is. Now E is by definition the work done per second per unit of current that passes in the circuit. So that in an element of time dt the electrical energy supplied by the source is $EIdt$. Now this

work is spent partly in heating the circuit; in this way a quantity of work is used up equal to $RI^2 dt$. The remainder of it is that part expended as mechanical work in the action between the circuit and the magnetic field; this part is $I\frac{dN}{dt}dt$.

Thus we have $EI\,dt = RI^2 dt + I\frac{dN}{dt}dt$.

$$\therefore \quad I = \frac{E - \frac{dN}{dt}}{R}.$$

Thus the effective E.M.F. in the circuit is no longer simply E, but is altered by the variation of induction through it. We say then that there is an E.M.F. induced in the circuit, and the measure of it is $-\frac{dN}{dt}$. Thus we have the rule,

The induced E.M.F. in a circuit is the rate of decrease of magnetic induction through the circuit.

This is a general proposition, of which we shall make several applications, and it applies to all cases of induced E.M.F. however the variation of induction through the circuit may be caused.

10. Suppose, for example, we have two circuits with a coefficient of mutual induction M. Let i be the current in one of them: then if i is varying the consequent induced E.M.F. in the other is $-M\frac{di}{dt}$. If M is varying in consequence of relative motion of the two circuits, the consequent induced E.M.F. is $-\frac{dM}{dt}i$.

When two circuits are used so that by varying currents in one, E.M.F.s are generated in the other, the two are called respectively the *primary* and *secondary circuits*, and the currents in them the *primary* and *secondary currents*.

Let A and B be two circuits with coefficient of mutual induction M, and let a steady current i_1 be maintained in A while the

current in B rises from o to i_2. Let R be the resistance of A. Then the E.M.F. in A is

$$E = Ri_1 + M\frac{di_2}{dt}.$$

Thus the work done to keep the current i_1 going in A while i_2 is set up in B, independently of that done in consequence of the resistance of A, is

$$\int Mi_1 \frac{di_2}{dt}\, dt = Mi_1 i_2.$$

This is also the expression for the work that must be done to keep i_1 going in A against the induction of B, if i_2 is kept steady in B while M increases from o to M; as, for instance, if B is brought up to position with its current i_2 from an infinite distance off. The same amount of work must also be done in B in this case to keep i_2 going against the inductive action of A.

Suppose the product Mi_2 to vary in any manner from the value $(Mi_2)_1$ to the value $(Mi_2)_2$, and suppose at the same time i_1 is kept steady in A. Then we have

$$E = Ri_1 + \frac{d}{dt}(Mi_2).$$

Thus the work done to keep i_1 steady in A against the inductive action of B is

$$i_1[(Mi_2)_2 - (Mi_2)_1].$$

11. If a given circuit is carrying a varying current it is producing a varying quantity of magnetic induction through itself. Thus there must be an induced E.M.F. in the circuit in consequence of the variation of the current that it carries. Now the quantity of induction through the circuit due to its current is generally proportional to its current. For unit current in the circuit let this quantity be L. L is called the *coefficient of self-induction* of the circuit: and we have for its definition,

DEF.: **THE COEFFICIENT OR SELF-INDUCTION of a circuit is the quantity of induction passing through it per unit current in it.**

L is also called the SELF-INDUCTANCE of the circuit.

L will depend on the form and size of the circuit, and on the magnetic susceptibilities of any neighbouring substances. Suppose we know the value of L for a circuit of a single turn, and have a circuit of n turns all occupying practically the same position as the single turn, then the value of L for this circuit will be obtained by multiplying that for a single turn by n^2, for we have now produced the same effect as if we had made the current n times as great, so that we have made the induction n times as great, and at the same time we have n times as many turns for the induction to pass through.

12. As in the case of the coefficient of mutual induction, it should be noticed that L will not be constant for a given circuit if some of the lines of induction due to a current carried by it pass through substances of variable susceptibility, as iron or steel. L will then vary with the current used, and can only be regarded as constant for such currents as produce very small magnetic intensity in the iron or steel.

13. Suppose a circuit with a coefficient of self-induction L to be carrying a varying current i. There is an induced E.M.F. in it equal to $-L\dfrac{di}{dt}$.

Thus, if there is an E.M.F. in the circuit, and it has resistance R, we have

$$E = Ri + L\frac{di}{dt}.$$

The work expended in starting the current i in the circuit, independently of that spent in consequence of the resistance, is

$$\int L\frac{di}{dt}i\,dt = \tfrac{1}{2}Li^2.$$

Also if the current i has been started in the circuit, the circuit then containing no E.M.F., the expression $\frac{1}{2}Li^2$ will represent the energy of the circuit. The circuit being now left to itself the current i will soon disappear, but we can show that the heat, produced in the circuit as the current falls off from i to o, is

$\frac{1}{2}Li^2$. For during the decrease of the current, we have the relation

$$Ri+L\frac{di}{dt}=0.$$

Thus $$Ri^2=-Li\frac{di}{dt}.$$

Therefore if t is the time that the current takes to change from the value i to 0, the measure of the heat generated, in joules, is

$$R\int_0^t i^2\,dt=-L\int_0^t i\frac{di}{dt}\,dt$$
$$=\tfrac{1}{2}Li^2.$$

14. Suppose we suddenly close the circuit containing the E. M. F. E, so as to start the current in it; the current will rise from 0 to its maximum value. Let this be i_0, so that $i_0=E/R$. We have, if i is the value of the current at any instant,

$$E=Ri+L\frac{di}{dt},$$

or, $$R\,dt=L\frac{R\,di}{E-Ri}.$$

$$\therefore\quad -Rt=L\log\frac{E-Ri}{C}$$
$$=L\log\frac{E-Ri}{E}.$$

$$\therefore\quad i=\frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right)$$
$$=i_0\left(1-e^{-\frac{Rt}{L}}\right).$$

Thus the current rises to its final value i_0 only after an infinite time; but in a very short time after starting it is only imperceptibly different from i_0.

Now suppose the E. M. F. E removed, and the current to fall from i_0 to zero. We have

$$Ri+L\frac{di}{dt}=0.$$

$$\therefore \quad t = -\frac{L}{R}\log\frac{i}{C}$$
$$= -\frac{L}{R}\log\frac{i}{i_0}.$$
$$\therefore \quad i = i_0 e^{-\frac{Rt}{L}}.$$

The current that passes after the E. M. F. has been suppressed is called the extra current of self-induction.

Thus the current does not becomes zero till after an infinite time; but in a very short time it is appreciably zero for most coils, for in most cases L/R, which is a quantity of the dimensions of a time, is very small.

15. The coefficient of self-induction of a long solenoid of length l with n turns per unit length is the quantity of induction across a cross-section of the solenoid multiplied by ln, the number of its turns. Thus if s is its cross-section, and it has no core, it is

$$4\pi n^2 ls.$$

If it has a core of cross-section s', and susceptibility κ, the coefficient of self-induction is

$$4\pi n^2 l(s+4\pi\kappa s').$$

16. Energy of any number of circuits. Suppose we have any number of electric circuits having coefficients of self-induction L_1, L_2, L_3,...; and coefficients of mutual induction M_{12}, M_{13}, M_{23},...; so that any two of these such as M_{pq}, M_{qp} are equal. Let us find the work that must be spent to start currents i_1, i_2, i_3,... in these circuits, neglecting their resistances. Let E_1, E_2, E_3,... be the values of the E. M. F.s acting in the circuits at any instant to start the currents.

We have the equations

$$E_1 = L_1\frac{di_1}{dt} + M_{12}\frac{di_2}{dt} + \dots,$$
$$E_2 = M_{12}\frac{di_1}{dt} + L_2\frac{di_2}{dt} + \dots,$$
$$\dots \quad \dots \quad \dots \quad \dots$$

And the energy is

$$T=\int(E_1 i_1+E_2 i_2+\dots)\,dt,$$

the limits of integration being taken from the time of starting to the time when the currents have risen to their full value.

Now the part of T involving L_1 is

$$\int L_1 i_1 \frac{di_1}{dt}\,dt=\tfrac{1}{2}L_1 i_1^2.$$

The part involving M_{12} is

$$\int M_{12}\left(i_1\frac{di_2}{dt}+i_2\frac{di_1}{dt}\right)dt=M_{12}i_1 i_2\,.$$

And we get similar values for all such terms.

Thus we have

$$\begin{aligned}T=\tfrac{1}{2}L_1 i_1^2+\tfrac{1}{2}L_2 i_2^2+\dots,\\ +M_{12}i_1 i_2+M_{13}i_1 i_3+\dots.\end{aligned}$$

This quantity is called by Maxwell the *electro-kinetic energy* of the system.

17. If we suppose the circuits to have any resistances R_1, R_2,... ; and if the system be left to itself after having the currents i_1, i_2,... started in it, that is if the E. M. F.s in the circuits be suppressed, the heat developed in the circuits before they cease will be T.

For we have the equations

$$\begin{aligned}R_1 i_1&=-L_1\frac{di_1}{dt}-M_{12}\frac{di_2}{dt}-\dots,\\ R_2 i_2&=-M_{12}\frac{di_1}{dt}-L_2\frac{di_2}{dt}-\dots,\\ &\dots\dots\dots\dots\end{aligned}$$

And the heat developed is

$$\int(R_1 i_1^2+R_2 i_2^2+\dots)\,dt,$$

the limits of integration being taken from the time when the currents have the values i_1, i_2,... to the time when they cease. Thus we get for the value of the integral the expression which has been written equal to T in the last article.

18. If we suppose the circuits to have no resistances, and allow any motions to take place among them after the currents have been started and the E. M. F.s removed, the work done among them by their mechanical actions on each other will be equal to the diminution of the electrokinetic energy.

Now let us suppose that during any motions of the circuits, the currents in them are kept constant by means of E. M. F.s included in them.

These motions will cause variations in the M's, and if we suppose the forms as well as the positions of the circuits to be changed, in the L's too. Let us denote the increments by δL_1, δM_{12},.... The consequent increment in T is

$$\delta T = \tfrac{1}{2}\,\delta L_1 i_1^2 + \tfrac{1}{2}\,\delta L_2 i_2^2 + \dots$$
$$+\,\delta M_{12} i_1 i_2 + \delta M_{13} i_1 i_3 + \dots.$$

Now if the increments of magnetic induction through the circuits are δN_1, δN_2,... the entire energy supplied by the E. M. F.s is

$$\begin{aligned} & i_1\,\delta N_1 + i_2\,\delta N_2 + \dots \\ &= i_1\,(\delta L_1 i_1 + \delta M_{12} i_2 + \dots), \\ &+ i_2\,(\delta M_{12} i_1 + \delta L_2 i_2 + \dots), \\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &= \delta L_1 i_1^2 + \delta L_2 i_2^2 + 2\,\delta M_{12} i_1 i_2 + \dots, \\ &= 2\,\delta T. \end{aligned}$$

Thus if while the currents are kept constant, mechanical work W is done among the circuits, at the same time the electrokinetic energy is *increased* by W, and the energy supplied by the E. M. F.s is $2\,W$.

If the motions are infinitesimal, the electro-magnetic forces will be the same whether the currents are kept constant or the E. M. F.s removed, so that the mechanical work done will be the same. Thus the increment of electrokinetic energy in one case is equal to the decrement in the other case.

19. We can get another useful expression for the electrokinetic energy of a system of linear circuits carrying currents. Suppose all the currents to rise simultaneously and uniformly from

zero to their full values. The magnetic induction at every point in the field will do so too; and will, at any instant, be the same fraction of its full value as any current is of *its* full value.

Now suppose that a current through whose circuit there is induction N is increased by the infinitesimal amount di. The energy spent in doing this is $N di$. Thus the entire energy spent as far as this circuit is concerned will be

$$\int N di.$$

But N is always proportional to i; so that the energy expended in this circuit is $\frac{1}{2}$ the product of the final values of N and i. And the whole electrokinetic energy of the system is $\frac{1}{2}$ *the sum of the products of each current and the quantity of induction through it.*

This investigation, of course, in supposing N always proportional to i, supposes the field to be everywhere of constant permeability, that is, that at every point the permeability is the same for all values of intensity, as for instance when the circuits are in air. If this is not the case we shall not get so simple an expression for the electrokinetic energy, but its value will in any case be given by the formula

$$\Sigma \int N di.$$

20. Maxwell supposed that the electrokinetic energy is associated with the lines of induction, and thus distributed throughout the entire field. We may consider the way in which it is distributed as follows.

Imagine a filament, of infinitesimal cross-section, of a circuit in the field, carrying an infinitesimal current i. The energy contributed by this is $\frac{1}{2}$ the product of i and the quantity of induction through it. Now take a closed tube of induction linked with the current i, of infinitesimal cross-section, and having throughout it a quantity of induction N. Thus the energy associated with, or in, this tube, as far as the current i is concerned, is $\frac{1}{2}\, i N$. It should be noticed that the tube of

induction may be, wholly or in part, contained in the conductor from which the filament carrying i has been chosen, or in any conductor of the field. Suppose all the currents, in infinitesimal filaments, surrounding this tube in the + sense, add up algebraically to I. Then the entire quantity of energy in the tube is $\frac{1}{2} IN$.

Now suppose at any point of the tube the permeability is μ, the magnetic intensity H, and the cross-section of the tube S. Then $N = \mu H S$.

Also integrating all along the tube we get

$$\int H ds = 4\pi I.$$

$$\therefore \quad 4\pi IN = \mu HS \int H ds$$

$$= \int \mu H^2 S ds,$$

since μHS is constant throughout the tube, and may therefore be taken under the sign of integration.

Thus the energy in the tube

$$\frac{1}{2} IN = \frac{1}{8\pi} \int \mu H^2 S ds,$$

or the energy per unit volume at any point in the field is

$$\frac{\mu}{8\pi} \cdot H^2.$$

21. To find the total quantity of electricity Q that passes round a circuit of resistance R by induction, let N_1 and N_2 be the initial and final values of the induction through the circuit; then

$$Q = \int i \, dt,$$

$$= \int \frac{E}{R} dt,$$

$$= -\int \frac{1}{R} \frac{dN}{dt} dt,$$

$$= \frac{N_1 - N_2}{R}.$$

Thus if the circuit has coefficient of self-induction L and the E. M. F. is suddenly suppressed when the current in the circuit is i, the quantity of electricity that passes after this is

$$\frac{iL}{R}.$$

If the current in a neighbouring circuit, which has with the given circuit coefficient of mutual induction M, is changed in value from i_1 to i_2, the quantity of electricity which passes round the given circuit is

$$\frac{M(i_1-i_2)}{R}.$$

22. Induction in a coil produced by Rotation.

Suppose we have a coil of n turns, all in one plane, and quite close together; let S be the area enclosed by one turn. Let the coil be in a magnetic field of strength H, and be rotated about an axis in its plane, which is perpendicular to the lines of force of this field, with a uniform angular velocity ω. Let us consider the E. M. F. induced in the coil at any instant by this motion.

Let θ be the angle which the plane of the coil makes at any instant with the position in which it is at right angles to the lines of force of the field. The quantity of magnetic induction enclosed by all the turns of the coil is $nSH\cos\theta$.

The induced E. M. F. will be the rate of decrease of this quantity; that is, it is $nSH\omega\sin\theta$.

Thus there is an alternating E. M. F. induced in the coil, which changes sign at the end of every half-turn.

If the coil is closed we will thus get a series of alternating currents through it.

Let us suppose that the coil is closed by means of a wire, to the ends of which its ends are connected by means of sliding contacts, so as to remain always in contact with them during the rotation of the coil. Let R be the entire resistance, and L the coefficient of self-induction of the circuit so formed.

The equation for the current is, (putting $\theta=\omega t$),

$$nSH\omega \sin \omega t = RI + L\frac{dI}{dt}.$$

Multiplying by $\frac{e^{\frac{Rt}{L}}}{L}$, and integrating, we get

$$e^{\frac{Rt}{L}} . I = \frac{1}{L}\, nSH\omega \int e^{\frac{Rt}{L}} \sin \omega t . dt,$$

$$= nSH\omega \frac{e^{\frac{Rt}{L}}\left(\frac{R}{L}\sin \omega t - \omega \cos \omega t\right)}{L\left(\frac{R^2}{L^2}+\omega^2\right)} + C.$$

Thus $$I = nSH\omega . \frac{\frac{R}{L}\sin \omega t - \omega \cos \omega t}{L\left(\frac{R^2}{L^2}+\omega^2\right)} + Ce^{-\frac{Rt}{L}}.$$

The last term, which diminishes as t increases, will have no appreciable effect on the value of I after a few turns, and may be neglected.

Thus $$I = nSH\omega \frac{R \sin \omega t - L\omega \cos \omega t}{R^2 + L^2\omega^2}.$$

Put $L\omega/R = \tan \phi$, and we have

$$I = \frac{nSH\omega \sin(\theta - \phi)}{\sqrt{R^2+L^2\omega^2}}.$$

Thus the strength of the current follows the same law as the E. M. F., but is not proportional to the E. M. F.

The effect of the self-induction of the circuit on the value of the current may be stated as follows.

The same current would be obtained as in the actual case, at every instant, if there were no self-induction, and

(1) the resistance of the circuit were made $\sqrt{R^2+L^2\omega^2}$,

(2) the coil were retarded in its rotation so as to be at the constant angle ϕ behind its position in the actual case.

In the same way we may suppose the coil to be acted upon

by a sinusoidal E.M.F., induced in any manner, of period T. The E. M. F. at time t may be given by the formula

$$E = E_0 \sin \frac{2\pi t}{T}.$$

And putting $\tan\phi = 2\pi L/RT$ we have for the current at any instant

$$I = \frac{E_0 \sin\left(\frac{2\pi t}{T} - \phi\right)}{\sqrt{R^2 + \frac{4\pi^2 L^2}{T^2}}}.$$

The expression in the denominator of this fraction is called the **IMPEDANCE** of the circuit.

23. General definition of Impedance. If an E. M. F., e, which is periodic but not necessarily sinusoidal, acts on a circuit, the impedance of the circuit is to be defined as follows. Let T be the period of the E. M. F., and therefore also of the current when the steady state has been attained. Take the integrals $\int_0^T e^2\,dt$, and $\int_0^T I^2\,dt$ over any whole period. The results are obviously independent of the limits of integration provided they differ, in each case, by T. The quantities

$$\sqrt{\frac{1}{T}\int_0^T e^2\,dt} \quad \text{and} \quad \sqrt{\frac{1}{T}\int_0^T I^2\,dt}$$

are called the effective E. M. F. and effective current, respectively.

The impedance is the quantity obtained by dividing the effective E. M. F. by the effective current, or it is

$$\sqrt{\frac{\int_0^T e^2\,dt}{\int_0^T I^2\,dt}}.$$

In the case of a sinusoidal current, from the values given

above for E and I, the value we should obtain for the impedance by this definition is obviously that already given,

$$\sqrt{R^2+\frac{4\pi^2L^2}{T^2}}.$$

24. Discharge Current. Suppose a conductor of capacity C is charged, and then connected to earth by means of a wire of resistance R and coefficient of self-induction L. Let Q be the charge, and E the potential of the conductor at any instant; I the current through the wire.

Then
$$Q = EC,$$
$$E = RI + L\frac{dI}{dt},$$
$$I = -\frac{dQ}{dt}.$$
$$\therefore \quad \frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{1}{CL}Q = 0.$$

Let m_1, m_2 be the roots of

$$m^2+\frac{R}{L}m+\frac{1}{CL}=0.$$

(1) Let m_1, m_2 be real.

Then we have

$$Q = A_1e^{m_1t}+A_2e^{m_2t};$$

and
$$\therefore \; I = -A_1m_1e^{m_1t}-A_2m_2e^{m_2t},$$
with the conditions

$$A_1+A_2 = Q_0, \text{ initial charge on conductor;}$$

and
$$A_1m_1+A_2m_2 = 0; \text{ since initial current is zero.}$$

(2) Let m_1, m_2 be imaginary, and equal to $a \pm \sqrt{-1}\,\beta$, say.

Then we have

$$Q = A_1e^{at}\sin(\beta t+A_2),$$
$$I = A_1e^{at}\{a\sin(\beta t+A_2)+\beta\cos(\beta t+A_2)\},$$

with the conditions

$$A_1 \sin A_2 = Q_0,$$
$$a \sin A_2 + \beta \cos A_2 = 0.$$

Hence we may put

$$I = A_1 (a \cos A_2 - \beta \sin A_2) e^{at} \sin \beta t.$$

Thus we see that the character of the discharge will depend on the nature of the roots m_1, m_2. It will be **oscillatory or not** according as these roots are imaginary or real; that is, according as

$$\frac{R^2}{L^2} \lessgtr \frac{4}{CL},$$

or according as

$$L \gtrless \frac{CR^2}{4}.$$

The electric oscillations in an oscillatory discharge are extremely rapid, and we cannot strictly apply to them the same equations as are used in the case of a current whose variations take place with moderate rapidity, as that induced in a rotating coil, for instance; we know, for example, that the current must have very different values at different points of the wire at the same instant, the passage of the electricity along the wire being of the nature of the passage of waves. The results arrived at by this method, then, must only be considered as first approximations.

25. When a variable current, whose value at a given instant is I, is flowing through a conductor, whose resistance and coefficient of self-induction are R and L, we have the ordinary equation for the P. D. at the terminals of the conductor

$$E = RI + L\frac{dI}{dt} \ldots (1).$$

Now if the current is made a very rapidly alternating one, and we express the P. D. at the terminals of the conductor as the sum of two terms as follows,

$$E = R'I + L'\frac{dI}{dt} \ldots (2),$$

it is found that R' is greater than R, the resistance as defined with reference to steady currents, and L' is less than L.

The general explanation of this may be given in the following way. Equation (1) is found on the supposition that the variations in the current take place in such a manner that we may suppose that at any instant it is uniformly distributed across the whole of any given section of the conductor. But when a current is very rapidly started or stopped in the conductor, the several filaments of which we may suppose the conductor to be formed, by acting inductively on each other, cause the elements of current to be formed as far away from each other as possible. Thus the currents are established chiefly in the outer parts of the conductor, and things take place practically in the same way as if an inner core were removed, and thus the electrical resistance increased. Also if this core were removed we would now have a conductor whose several filaments are on the whole more separated from each other, and which would therefore have a smaller coefficient of self-induction.

As an example of the effect produced, we may notice the following case. In a soft iron wire of length 160 cms. and diameter 3·3 mm. with an alternating current of complete period 1/1050 sec. the mean value of R' was $1{\cdot}84\,R$.

26. Suppose we have two coils wound so that all the lines of induction through either due to a current in it pass through the other. This may practically be arranged by putting the coils as close as possible to each other, or by winding them on the same ring of very soft iron, as a core, so that the ring threads through every turn of each coil, and practically all the lines of induction due to the coils lie in the substance of the ring.

Further, suppose that the resistances of the coils are so small, and the inductances and rates of variation of current so large, that in the equations of E.M.F. we may neglect the terms depending on the resistances.

If we have a variable E.M.F. given to the primary coil, supplied at its terminals, let us consider the E.M.F. generated

in the secondary; or given off at its terminals, the resistance being negligible, if it is joined to an outside circuit containing resistance.

This is the problem that occurs in the case of electrical transformers.

Suppose the numbers of turns in the two coils are n_1 and n_2.

Let L, N be the self-inductances, and M the mutual inductance of the coils. Then since for any current in the first coil the same quantity of induction passes through both coils, L and M are proportional to n_1 times and n_2 times this quantity of induction. Thus

$$L : M = n_1 : n_2.$$

Similarly

$$M : N = n_1 : n_2.$$

Now let E_1, E_2 be the E. M. F.s at the terminals of the coils at any instant, and i_1, i_2 the currents in them.

Then we have approximately, neglecting the resistances, the equations,

$$E_1 = L\frac{di_1}{dt} + M\frac{di_2}{dt},$$

$$E_2 = M\frac{di_1}{dt} + N\frac{di_2}{dt}.$$

Thus, using the above relations between L, M, N,

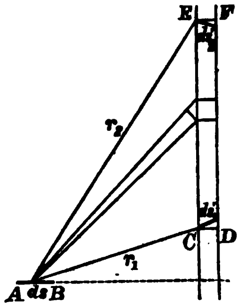
$$\frac{E_1}{E_1} = \frac{n_1}{n_2}.$$

Thus in this case the E. M. F. is practically changed in the ratio of the number of turns in the primary to the number in the secondary.

27. To find the mutual potential of two closed electric circuits carrying currents i and i'.

Let us denote the two circuits by the letters s and s'. Then the required mutual potential is the product of i' and the quantity of magnetic induction due to s passing through s' in the positive direction, with its sign changed.

Let AB denote an element ds of s. And let a series of indefinitely near planes be drawn at right angles to AB, the distance between two consecutive planes being t. And let these two cut off elements ds'_1, ds'_2 from s'; these elements being at distances r_1 and r_2 from ds; and r_1 and r_2 making angles θ_1 and θ_2 with positive direction of ds. Let the positive directions of ds, ds'_1 in the figure be to the right, and of ds'_2 to the left.



Now let us consider the quantity of magnetic induction due to ds enclosed between ds'_1, ds'_2 and the two planes. The lines of force due to ds are circles about ds as axis. So that if r_1 and ds'_1 are turned about ds as an axis till r_1 comes into the plane of r_2 and ds, ds'_1 will during the motion cut no lines of force, and the quantity of magnetic induction required is that now enclosed between ds'_1, ds'_2 and the two planes. But the lines of force all cutting the present plane of r_1 and r_2 at right angles, this will be the quantity of induction passing across the portion $CDFE$ of this plane enclosed between the two planes CE, DF and two straight lines CD, EF at right angles to them at the extremities of r_1 and r_2.

Now the magnetic intensity at an element of this area at a distance r from ds, and such that r makes an angle θ with ds, is

$$i\,ds\,\frac{\sin\theta}{r^2}.$$

And the area of this element is

$$\frac{dr}{\sin\theta}\,.\,t.$$

Thus the quantity of induction across $CDFE$ due to ds is

$$i\,ds\,t\int_{r_1}^{r_2}\frac{dr}{r^2} = i\,ds\,t\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

Now if the positive directions of ds'_1 and ds'_2 make angles ϵ_1 and ϵ_2 with that of ds we have $t = ds'_1 \cos\epsilon_1 = -ds'_2 \cos\epsilon_2$.

Thus the above expression is

$$-i\,ds\left(\frac{ds'_1 \cos \epsilon_1}{r_1} + \frac{ds'_2 \cos \epsilon}{r_2}\right).$$

And considering all such elements as ds'_1 and ds'_2 the entire quantity of magnetic induction due to ds enclosed by s' in the positive direction is

$$i\,ds\int\frac{\cos \epsilon}{r}\,ds'.$$

And the entire quantity of magnetic induction, or number of lines of force, due to s enclosed by s' in the positive direction is

$$i\iint\frac{\cos \epsilon}{r}\,ds\,ds'.$$

Thus the mutual potential of the two circuits is

$$-ii'\iint\frac{\cos \epsilon}{r}\,ds\,ds'.$$

Therefore the coefficient of mutual induction of two circuits is

$$\iint\frac{\cos \epsilon}{r}\,ds\,ds'.$$

This expression is also called the *mutual potential* of the two *circuits.*

The same double integral taken round one circuit is the coefficient of self-induction of the circuit.

28. Vector Potential of Magnetic Induction.

The quantity of induction through any closed circuit is expressed by the surface integral

$$\iint N\,dS$$

taken over any closed surface bounded by the circuit. Now the same quantity may be expressed by means of a line integral taken round the circuit. Suppose J is some quantity having at every point of the field a definite direction and magnitude, ϵ denoting the angle between J and ds, an element of the circuit, and J being such that the line integral

$$\int J \cos \epsilon\,ds$$

taken round the circuit has the same value as the above surface integral. The quantity J was introduced by Maxwell, and called by him the *Vector Potential of Magnetic Induction.* We shall now see how J must be related to the induction that it may satisfy the required condition of making the above two integrals equal.

From the circuit cut out an infinitesimal length du, and move this parallel to itself through a distance dv in a direction at right angles to its old position. Join up the ends of du to their old positions on the circuit by two lengths each equal to dv.

Then, if N is the component of magnetic induction at right angles to the rectangle du, dv, the quantity of induction so added to the circuit is $N\,du\,dv$.

Again, let J_u, J_v be the components of J along du and dv, in the positive sense round the circuit. Then the addition to the quantity of induction through the circuit is got from the line integral by taking in the terms contributed along dv, du, dv, and subtracting the term that arose from the old position of du: or, what comes to the same thing, taking the terms arising from the four sides of the rectangle in order, paying due regard to the signs. These terms may be written as

$$J_v \,.\, dv,\ \left(J_u + \frac{dJ_u}{dv}\,dv\right)du,$$

$$-\left(J_v + \frac{dJ_v}{du}\,du\right)dv,\ -J_u \,.\, du.$$

Their sum is

$$\left(\frac{dJ_u}{dv} - \frac{dJ_v}{du}\right) du\,dv.$$

And this must be equal to $N\,du\,dv$.

Thus we have

$$N = \frac{dJ_u}{dv} - \frac{dJ_v}{du}.$$

The components of J at any point of the field must be connected with those of the magnetic induction by relations of this sort. And conversely, if a function J is found to satisfy these relations

at all points of the field it will fulfil the necessary condition of making the integrals

$$\iint N dS \text{ and } \int J ds$$

equal for any infinitesimal rectangle, and therefore for any circuit whatever.

Taking F, G, H for the components of J at any point, and a, b, c for those of the induction, we have the relations

$$a = \frac{dH}{dy} - \frac{dG}{dz},$$

$$b = \frac{dF}{dz} - \frac{dH}{dx},$$

$$c = \frac{dG}{dx} - \frac{dF}{dy}.$$

29. An expression for the electrokinetic energy of a system of currents flowing in conductors of any demensions may be found in terms of F, G, H. We have for the energy the expression

$$\tfrac{1}{2}\Sigma(iN),$$

$$\text{or } \tfrac{1}{2}\Sigma\left[i\int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds\right],$$

$$\text{or } \tfrac{1}{2}\Sigma\left[\int i\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds\right],$$

since i is constant round any filamental circuit and may therefore be taken under the integral.

i denotes, as in Art. 20, the infinitesimal current in any filamental circuit.

Now suppose that at any point the current per unit area, drawn across the direction of its flow, is q; and let u, v, w be the components of q.

Let S denote the area of cross-section of a filamental circuit. Then $q = i/S$;

$$\therefore \quad i\frac{dx}{ds}, \quad i\frac{dy}{ds}, \quad i\frac{dz}{ds} \text{ are equal to}$$

$$uS, \quad vS, \quad wS.$$

Substituting the values in the integral and writing instead of $S\,ds$ the expression for the element of volume $dx\,dy\,dz$, we get for the electrokinetic energy the expression

$$\iiint (Fu + Gv + Hw)\,dx\,dy\,dz$$

taken throughout the field.

30. The following method of treating magnetic tubes of induction is due to Hopkinson.

Suppose we have a tube of induction, the cross-section at any place, permeability (considered constant throughout the cross-section), intensity and induction being denoted by s, μ, H and B. Let dl denote an element of length of the tube. Then for the line integral of intensity round the tube we have

$$\begin{aligned}\int H\,dl &= \int \frac{B\,dl}{\mu}\\ &= \int \frac{Bs\,dl}{\mu s}\\ &= Bs\int \frac{dl}{\mu s},\end{aligned}$$

$\because$ Bs is constant throughout the tube. $\int H\,dl$ is called the magneto-motive force of the circuit. Bs is the quantity of induction through the tube; or, as it is sometimes called, the flow, or flux, of induction in the tube. $1/\mu$ may be called the specific magnetic resistance, and denoted by ρ; so that $\int \frac{dl}{\mu s} = \int \rho \frac{dl}{s}$ is the magnetic resistance of the circuit which the tube forms. Thus we have the formula

magneto-motive force $=$ *flow of induction* $\times$ *magnetic resistance*;

precisely analogous to the electrical formula

electromotive force $=$ *current* $\times$ *electrical resistance.*

If i is the algebraic sum of all the currents interlinking the tube in the positive sense the magnetomotive force is $4\pi i$; so that if no current interlinks the tube, the magneto-motive force is zero.

In the case of a substance having retentiveness we may get a flow of induction, due to residual magnetism, when the magneto-motive force is zero, or even of opposite sign from the flow. The explanation of this is that, in the above formula, the magnetic resistance is then zero or negative. For, by definition, ρ is, at any point, equal to H/B; and, in such a substance, H may be zero while B is not, or H and B may have opposite signs. Thus the values of ρ throughout the tube may be such as to make the integral expressing the magnetic resistance zero or negative.

We may take for the tube of force forming the *magnetic circuit* any material into, or out from, which no lines of induction pass; that is, across the sides of which there is no *magnetic leakage.*

Consider a uniform circular ring, length l, cross-section s, wound round uniformly with n turns of wire, carrying current i. The flow of induction will be

$$Bs = \frac{4\pi ni}{\dfrac{l}{\mu s}}$$

$$= 4\pi\mu sni/l.$$

Next, suppose the ring is slit across by a gap of breadth δl, so narrow in comparison with the thickness of the ring that we may suppose the lines of force not to bulge out at the gap; that is, the magnetic circuit is of practically the same breadth throughout. The magnetic resistance is now

$$\frac{l-\delta l}{\mu s} + \frac{\delta l}{s}$$

$$= \frac{l+(\mu-1)\,\delta l}{\mu s}.$$

Thus it is the same as if the ring were increased by a length $(\mu-1)\,\delta l$, and the flow of induction is now

$$\frac{4\pi\mu sni}{l+(\mu-1)\,\delta l}.$$

CHAPTER VIII.

GALVANOMETERS.

1. We have already noticed a few of the commonest forms of galvanometers and their chief uses, we shall now go on to consider them more in detail.

Sine Galvanometer. Suppose we have a galvanometer of any form with a needle of any size; and suppose we can rotate its coils about a vertical axis and have some means of measuring the angle through which they have been rotated from the magnetic meridian. Let M be the magnetic moment of the needle, and G the moment of the forces with which unit current in the coils acts on the needle when both coils and needle are in the same vertical plane. Let H be the horizontal component of the earth's magnetic intensity. Suppose a current C to be passed in the coils causing a deflexion of the needle. Now turn the coils so as to follow up the needle, and when they are both again in the same vertical plane let θ be the angle through which they have been turned. The needle is in equilibrium under the action of the two couples whose moments are CG and $MH \sin\theta$.

$$\therefore \quad C = \frac{MH}{G} \sin\theta \,.$$

2. The ordinary method of observing the deflexion of the needle of a sensitive galvanometer is as follows. To the needle is attached a small plane mirror. A divided scale is placed horizontally at some distance from the galvanometer. A beam of light is made to pass through a slit, just under the scale, in

which is a vertical wire, and to fall on the mirror after passing at right angles to the scale. A lens is placed between the slit and the mirror so as to form an image of the wire on the scale after reflexion of the light at the mirror. Sometimes the mirror is made concave, so that a lens is unnecessary: then the wire is placed at the centre of the mirror. When the needle is in its position of equilibrium, there being no current in the galvanometer, the image of the wire is formed just over the wire itself, that is, at the foot of the perpendicular from the mirror on the scale. Very frequently, in using this instrument, the thing to be observed is simply whether there is a current or not; but if we wish to determine the deflexion of the needle, let l be the distance of the scale from the needle, d the distance from the zero reading at which the image of the wire is formed. Let θ be the deflexion of the needle; then the deflexion of the reflected ray is 2θ. Then $\tan 2\theta = d/l$. From this θ can be determined.

To increase the sensitiveness of a sensitive galvanometer it frequently has attached to it a magnet, called a compensating magnet, which can be adjusted so as to diminish the action of the earth's magnetism on the needle, and to allow a larger deflexion to be obtained for a given current.

3. Winding of sensitive galvanometer. In winding a sensitive galvanometer we wish to arrange so that the turns may produce the greatest possible magnetic effect on the needle. We should then get the wire as close as possible to the needle. Now a certain amount of space must be left for the needle to swing freely in, which must not be occupied by the wire: let us consider what is the best possible winding for the wire outside this space. Suppose the windings to be all circles about the horizontal magnetic axis of the needle in its position of equilibrium, as a common axis. Consider a length l of the wire wound in a circle of radius $r \sin \theta$ with its centre at a distance $r \cos \theta$ from the centre of the needle. Let i be the current flowing in this wire. Then the magnetic intensity produced by the portion of wire we are considering

resolved horizontally and at right angles to the axis of the needle is

$$\frac{il \sin\theta}{r^2}.$$

If then the length l is wound anywhere on the boundary of a surface of revolution for which $\sin\theta/r^2$ is constant, it will, when carrying a given current, produce a constant effect on the needle. If wound outside this surface its effect will be diminished. The wire which is wound on the galvanometer should then, to produce the greatest possible effect, just fill up the volume included between the space left for the needle and the surface $r^2 = a^2 \sin\theta$, where a is some constant.

4. Ballistic Galvanometer. This is an instrument in which the needle is made to have a considerable moment of inertia; it is sometimes made of a magnetized sphere of steel. It is used when we wish to read, not the steady deflexion of the needle, but the extent of the swing produced in it by a transitory current. With a needle having a large moment of inertia the entire transitory current may be assumed to have passed before the needle has moved sensibly from its position of equilibrium, and as it moves slowly the extent of the swing will be easier to read than that of a light, quickly-swinging needle.

Let M be the magnetic moment of the magnet, A its moment of inertia, H the horizontal component of the earth's magnetic intensity, G the constant of the galvanometer. Suppose a transitory current passed through the galvanometer, Q being the entire quantity of electricity that passes, and let α be the swing of the needle. If i is the current at any instant, we have

$$Q = \int i\,dt.$$

Let ω be the angular velocity with which the needle starts The angular momentum communicated to it is

$$\int MGi\,dt = MGQ.$$

$$\therefore \quad A\omega = MGQ.$$

Thus the kinetic energy with which the needle starts is

$$\tfrac{1}{2} A\, \omega^2 = \frac{M^2 G^2 Q^2}{2A}.$$

And the work done against the magnetic forces before the needle comes to rest is

$$\int_0^a MH \sin\theta\, d\theta = MH(1 - \cos a) = 2\, MH \sin^2\frac{a}{2};$$

$$\therefore \quad \frac{M^2 G^2 Q^2}{2A} = 2\, MH \sin^2\frac{a}{2};$$

$$\therefore \quad Q = \frac{2}{G}\sqrt{\frac{AH}{M}} \sin\frac{a}{2}.$$

Now if the needle is set to make small oscillations under the action of the earth's field alone, its equation of motion is

$$A\frac{d^2\theta}{dt^2} + MH\sin\theta = 0,$$

so that the time of a complete oscillation is

$$T = 2\pi\sqrt{\frac{A}{MH}};$$

$$\therefore \quad Q = \frac{H}{G}\,\frac{T}{\pi}\sin\frac{a}{2}$$

$$= \frac{H}{G}\,\frac{T}{2\pi}\,a,$$

when the deflexion is small.

5. Damping of Galvanometer. In some galvanometers, in order to bring the needle to rest more quickly, a resistance is offered to its motion; so that its oscillations rapidly become smaller and smaller, and the needle soon settles down at its position of equilibrium. Such a galvanometer is said to be damped. A common way of damping a galvanometer is to have a mass of copper close to the needle, so that as the needle swings it induces currents in the copper, and these retard the motion of the needle. Sometimes the needle is suspended inside a sphere of copper which has a cavity hollowed out inside it for the

needle to swing in. The currents induced in the copper in any position of the needle are proportional to the velocity of the needle, and the retarding effect on the needle is proportional to the intensity of the currents, so that it is proportional to the velocity of the needle, for a given position of the needle. If we suppose the oscillations to be small, we may always suppose the retarding effect to be proportional to the velocity of swing of the needle.

Let the deflexion of the needle at any instant from its position of equilibrium be θ. Then the equation of motion may be written

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \omega^2\theta = 0,$$

the factor k depending on the damping, and ω^2 on the restoring couple.

To solve this equation let us put

$$\theta = e^{-kt}.x,$$

$$\frac{d\theta}{dt} = e^{-kt}\left(\frac{dx}{dt} - kx\right),$$

$$\frac{d^2\theta}{dt^2} = e^{-kt}\left(\frac{d^2x}{dt^2} - 2k\frac{dx}{dt} + k^2x\right).$$

Thus we get

$$\frac{d^2x}{dt^2} + (\omega^2 - k^2)\,x = 0.$$

From which

$$x = C\sin\left(\sqrt{\omega^2 - k^2}\,t + C'\right);$$

$$\therefore\quad \theta = Ce^{-kt}\sin\left(\sqrt{\omega^2 - k^2}\,t + C'\right);$$

if ω is $> k$.

Or
$$x = Ce^{\sqrt{k^2-\omega^2}\,t} + C'e^{-\sqrt{k^2-\omega^2}\,t};$$

$$\therefore\quad \theta = Ce^{-(k-\sqrt{k^2-\omega^2})t} + C'e^{-(k+\sqrt{k^2-\omega^2})t};$$

if k is $> \omega$.

In this last case the motion is not oscillatory; the needle can only go through the position of equilibrium for one value of t;

after which it reaches a point of maximum elongation, and then continually approaches the position of equilibrium, only reaching it after an infinite time.

Similar results would be obtained if $\omega = k$, the solution for which case is

$$\theta = e^{-kt}(C + C't).$$

Let us consider the case in which ω is $> k$. The equation shows that we get isochronous oscillations of which the complete period is

$$T = \frac{2\pi}{\sqrt{\omega^2 - k^2}}.$$

But the amplitudes of these oscillations diminish in geometrical progression. If the value of a given half-amplitude is

$$Ce^{-kt},$$

that of the next one will be

$$Ce^{-k\left(t + \frac{\pi}{\sqrt{\omega^2 - k^2}}\right)}.$$

Thus the ratio of each half-amplitude to the next is $e^{\frac{k\pi}{\sqrt{\omega^2 - k^2}}}$. The logarithm of this ratio, $k\pi/\sqrt{\omega^2 - k^2}$, is called the *logarithmic decrement.* Let us denote it by λ. The common logarithmic decrement L is found by observing an elongation, and then another after the body has made a considerable number of vibrations. Then

$$\lambda = L \log_e 10.$$

If T_0 is the period of a complete oscillation without damping,

$$T_0 = \frac{2\pi}{\omega}.$$

$$\therefore \quad \frac{T_0}{T} = \frac{\sqrt{\omega^2 - k^2}}{\omega} = \frac{\pi}{\sqrt{\pi^2 + \lambda^2}}.$$

Suppose we have a damped ballistic galvanometer, and a quantity of electricity Q passes through it. Let a be the observed deflexion, a_0 that which we would obtain with no damping.

The position of the needle at any instant, measuring the time from the instant when it starts, is given by

$$\theta = Ce^{-kt} \sin \sqrt{\omega^2 - k^2}\, t.$$

Thus to find the maximum elongation, we have

$$0 = \frac{d\theta}{dt} = Ce^{-kt}\left(\sqrt{\omega^2 - k^2} \cos \sqrt{\omega^2 - k^2}\, t - k \sin \sqrt{\omega^2 - k^2}\, t\right);$$

$$\tan \sqrt{\omega^2 - k^2}\, t = \frac{\sqrt{\omega^2 - k^2}}{k}.$$

And if v is the initial angular velocity,

$$v = C\sqrt{\omega^2 - k^2};$$

$$\therefore \quad a = \frac{v}{\sqrt{\omega^2 - k^2}} e^{-\frac{k}{\sqrt{\omega^2 - k^2}} \tan^{-1} \frac{\sqrt{\omega^2 - k^2}}{k}} \frac{\sqrt{\omega^2 - k^2}}{\omega}.$$

And $\quad a_0 = \dfrac{v}{\omega}.$

$$\therefore \quad a_0 = a \,.\, e^{\frac{\lambda}{\pi} \tan^{-1} \frac{\pi}{\lambda}}.$$

Now $\quad Q = \dfrac{H}{G} \dfrac{T_0}{2\pi} a_0.$

Thus $\quad Q = \dfrac{H}{G} \dfrac{T}{2\sqrt{\pi^2 + \lambda^2}} e^{\frac{\lambda}{\pi} \tan^{-1} \frac{\pi}{\lambda}} a.$

If λ is small we have $\tan^{-1} \pi/\lambda = \pi/2$ nearly.

Therefore $\quad Q = \dfrac{H}{G} \dfrac{T}{2\pi}\left(1 + \dfrac{\lambda}{2}\right) a.$

Suppose there is very little damping, so that λ is very small. Let $1 + \rho$ be the ratio of each half-amplitude to the next. Then ρ is very small. And we have

$$\lambda = \rho - \frac{\rho^2}{2} + \dots$$

$$= \rho \text{ very nearly.}$$

Thus $\quad Q = \dfrac{H}{G} \cdot \dfrac{T}{2\pi}\left(1 + \dfrac{\rho}{2}\right) a.$

6. To find a more exact formula we may notice that a and a_0 should more strictly be found from the equations

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \omega^2 \sin\theta = 0,$$

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0.$$

Now since k and $d\theta/dt$ are both small, by using the approximate equations with θ for $\sin\theta$, we nearly change the angular acceleration in each case in the same ratio. And so the values found for the maximum elongation from these equations are nearly changed in the same ratio. Thus if a and a_0 were found from the more exact equations we should still have very approximately

$$a_0 = a \,.\, e^{\frac{\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}};$$

or

$$a_0 = a\left(1 + \frac{\lambda}{2}\right).$$

Again, the exact value for Q is

$$\frac{H}{G}\frac{T_0}{\pi}\sin\frac{a_0}{2}$$

$$= \frac{H}{G}\frac{T}{\sqrt{\pi^2+\lambda^2}}\sin\frac{a_0}{2}.$$

Now

$$\sin\frac{a_0}{2} = \sin\left\{\frac{a}{2}\left(1 + \frac{\lambda}{2}\right)\right\}.$$

And on expanding this is seen to be equal to

$$\sin\frac{a}{2}\left(1 + \frac{\lambda}{2}\right),$$

as far as small quantities of the third order are concerned, a and λ being both small.

Thus

$$Q = \frac{H}{G}\frac{T}{\pi}\sin\frac{a}{2}\left(1 + \frac{\lambda}{2}\right).$$

7. In using a reflecting galvanometer when a steady deflexion, or a throw, of the needle is obtained, the corresponding current,

or quantity of electricity, may be taken as proportional to the deflexion, or throw, as measured on the scale. If more accurate results are required we may proceed as follows.

Let θ be the angular displacement of the needle, d the displacement of the spot of light on the scale, l the distance of the mirror from the scale. Then we may require $\tan\theta$ or $\sin\theta/2$, and we have $\tan 2\theta = d/l$. Writing the equations approximately, we have

$$2\theta = \tan 2\theta - \frac{\tan^3 2\theta}{3}$$

$$\theta = \frac{d}{2l} - \frac{d^3}{6l^3}.$$

$$\tan\theta = \theta + \frac{\theta^3}{3}$$

$$= \frac{d}{2l} - \frac{d^3}{6l^3} + \frac{d^3}{24l^3}$$

$$= \frac{d}{2l}\left(1 - \frac{d^2}{4l^2}\right).$$

$$\sin\frac{\theta}{2} = \frac{\theta}{2} - \frac{\theta^3}{48}$$

$$= \frac{d}{4l} - \frac{d^3}{12l^3} - \frac{d^3}{48.8l^3}$$

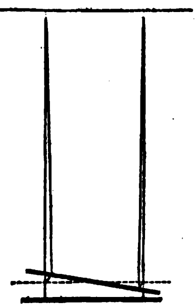
$$= \frac{d}{4l}\left(1 - \frac{11}{32}\frac{d^2}{l^2}\right).$$

The proportional errors in these formulae are clearly of the magnitudes of d^4/l^4, since the full series in the brackets only contain even powers of d/l, for when d is changed in sign and not magnitude so are $\tan\theta$ and $\sin\theta/2$.

8. Bifilar Suspension. A bifilar suspension is used in many electrical and magnetic instruments; and we shall here give the theory of it.

A weight P is suspended by two threads of equal length l, having their upper ends on the same horizontal line, and which in the position of equilibrium are parallel to each other, and at a distance apart d. If the body is turned through an angle θ about

a vertical axis from its position of equilibrium its centre of gravity is raised, and it tends to return to its original position. To find the moment of the couple acting on the body to produce this displacement. Let this moment be M. Suppose the centre of gravity of the body, which we may consider to be vertically below a point midway between the attachments of the strings, to be raised through a distance x. Let each of the strings be deviated from its vertical position by an angle ϕ.



Then

$$x = l(1 - \cos\phi),$$

$$l \sin\phi = d \sin\frac{\theta}{2},$$

$$\therefore \quad x = l - \sqrt{l^2 - d^2 \sin^2\frac{\theta}{2}}.$$

For the position of equilibrium

$$M d\theta = Pg\,dx,$$

$$= \frac{Pgd^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{2\sqrt{l^2 - d^2\sin^2\frac{\theta}{2}}},$$

$$\therefore \quad M = \frac{Pgd^2 \sin\theta}{4\sqrt{l^2 - d^2\sin^2\frac{\theta}{2}}}.$$

Or since d is made very small in comparison with l, and θ is taken small too,

$$M = \frac{Pgd^2 \sin\theta}{4l}, \text{ very nearly.}$$

9. Weber's Electrodynamometer. In this instrument a small coil is suspended by a bifilar suspension, consisting of two wires, within a large one which is fixed in the plane of the magnetic meridian. Let G be the magnetic intensity produced by the large coil at its centre, where the small one is suspended,

for unit current flowing in it. Let g be the magnetic moment of the small coil for unit current flowing in it. The small coil hangs from a torsion-head which may be turned through any required angle. The current is led to this coil by means of the wires forming the suspension. Let the moment of torsion due to the suspension, when the angle of torsion is θ, be $F\sin\theta$. Suppose we have currents i_1, i_2 in the fixed and suspended coils. And let θ be the angle through which the suspended coil is deflected. This angle is observed by means of a mirror attached to the coil and reflecting a beam of light. We have the equation,

$$i_1 i_2 Gg\cos\theta - i_2 gH\sin\theta - F\sin\theta = 0.$$

To correct for errors of adjustment of the coils, let us suppose that the plane of the fixed coil makes an angle α, and the axis of the suspended coil an angle $\alpha+\beta$ with the magnetic meridian. Suppose we pass the current i_2 in the suspended coil, and then reverse its direction, and obtain the corresponding deflexions θ_1, θ_2 on opposite sides of its position of equilibrium. We get the equations,

$$i_1 i_2 Gg\cos(\theta_1+\beta) = i_2 gH\sin(\theta_1+\alpha+\beta) + F\sin\theta_1, \qquad (1)$$
$$i_1 i_2 Gg\cos(\theta_2-\beta) = -i_2 gH\sin(\theta_2-\alpha-\beta) + F\sin\theta_2. \quad (2)$$

In the same way if by passing the current i_1 in the opposite directions through the fixed coil, we get for the two directions of i_2 in the suspended coil the deflexions θ_3, θ_4 on opposite sides of the position of equilibrium, we have the equations,

$$i_1 i_2 Gg\cos(\theta_3-\beta) = i_2 gH\sin(\theta_3-\alpha-\beta) + F\sin\theta_3, \qquad (3)$$
$$i_1 i_2 Gg\cos(\theta_4+\beta) = -i_2 gH\sin(\theta_4+\alpha+\beta) + F\sin\theta_4. \quad (4)$$

From (1) and (3) we get

$$i_1 i_2 Gg = i_2 gH\tan\frac{\theta_1+\theta_3}{2}\cdot\frac{\cos\left(\frac{\theta_1-\theta_3}{2}+\alpha+\beta\right)}{\cos\left(\frac{\theta_1-\theta_3}{2}+\beta\right)}$$
$$+F\tan\frac{\theta_1+\theta_3}{2}\cdot\frac{\cos\frac{\theta_1-\theta_3}{2}}{\cos\left(\frac{\theta_1-\theta_3}{2}+\beta\right)}.$$

Now α and β are very small angles; and θ_1, θ_2, θ_3, θ_4 are nearly equal. Thus neglecting magnitudes above the first order of small quantities, we get from this equation,

$$i_1 i_2 Gg = i_2 gH \tan\frac{\theta_1+\theta_3}{2} + F\tan\frac{\theta_1+\theta_3}{2}.$$

In a similar way from equations (2) and (4), we get

$$i_1 i_2 Gg = -i_2 gH \tan\frac{\theta_2+\theta_4}{2} + F\tan\frac{\theta_2+\theta_4}{2}.$$

Now suppose that $i_2 gH$ is small compared with F, then at the same time the deflexions are nearly equal. Thus approximately

$$i_1 i_2 = \frac{F}{2Gg}\left(\tan\frac{\theta_1+\theta_3}{2} + \tan\frac{\theta_2+\theta_4}{2}\right),$$
$$= \frac{F}{Gg}\tan\frac{\theta_1+\theta_2+\theta_3+\theta_4}{4}, \text{ nearly.}$$

If the same current i is sent through the two coils, and θ is the mean value of the deflexions,

$$i^2 = \frac{F}{Gg}\tan\theta.$$

This method of using the electrodynamometer is called the Method of Tangents.

Another method of using it is to bring back the suspended coil to its position of equilibrium, with its axis in the magnetic meridian, while the current is passing. Then we have, if the adjustments are accurate,

$$i_1 i_2 = \frac{F}{Gg}\sin\theta.$$

Or with the same current in both coils,

$$i^2 = \frac{F}{Gg}\sin\theta.$$

This is called the Method of Sines.

If the current in each coil has its direction changed the deflexion is unaffected. An electrodynamometer is consequently a suitable instrument for measuring alternating currents.

CHAPTER IX.

Determination of the Absolute Value of a given Electrical Resistance.

1. We have seen how an electrical resistance may be measured in terms of a given set of standard resistances. This operation may be called *Comparison of Resistances*: with suitable apparatus it is capable of being done with tolerable ease and a considerable degree of accuracy. Thus with a Post Office resistance box, in which the values of the resistances of the coils are given in terms of the B. A. unit of resistance, we can find to a small per-centage the value of a given electrical resistance in B. A. units.

We have now to consider methods of determining the value of a given resistance in terms of the absolute unit of resistance, or the ohm, according to the definition already given of this unit. The object of this is to determine in terms of some easily producible standard the value of the ohm. The plan adopted has been to compare the resistance measured in ohms with a column of pure mercury of known dimensions and at 0° C., and to calculate from the value of its resistance in ohms the length of a column of pure mercury at 0° C. having a cross-section of one square millimetre, which has a resistance of one ohm.

When this has been done, copies of the ohm consisting of columns of mercury, with dimensions adjusted to have the required resistance at a fixed temperature, or conductors of

other metals compared electrically with these, may be made and kept as absolute units of resistance.

2. The methods which have been used to determine the absolute value of a given resistance may be classed under two general heads. They depend respectively on the laws of Joule and Ohm.

In the first, the resistance of a conductor is calculated from the amount of heat generated in it by a known current in a given time, according to Joule's law which has for its expression the equation

$$JH = C^2Rt,$$

where the heat H is measured in calories, and J is the mechanical equivalent of heat.

In the second, the resistance of a conductor is calculated from the current produced in a conductor by a known E. M. F. according to Ohm's law which has for its expression the equation

$$E = CR.$$

In this case the E. M. F. must be due to induction, as these are the only E. M. F.s that can be calculated directly in absolute units.

We shall now describe some of the principal methods that have been used to determine the absolute value of a resistance.

3. (1) **Joule's Calorimetric Method.** A current C whose strength was measured by means of a tangent galvanometer was passed through the coil of resistance R contained with water in a calorimeter. The heat, H, in calories, developed in the calorimeter in a certain time, t seconds, was determined. Then if J is the mechanical equivalent of heat we have, to determine R, the equation

$$JH = C^2Rt.$$

This method has been used by Joule and other experimenters. There is this objection to it; that the value of J is not known sufficiently well to make it an accurate method. In fact the value of the ohm is probably known, from other methods, much better than J is known. This would therefore be a more suit-

able method for determining the value of J from the value in ohms of the resistance of the coil used in the calorimeter. At the same time it should be mentioned that the value which Joule determined for the ohm by this method is now known to be much more accurate than many determinations made before and since by means of methods not involving the same difficulties.

4. (2) Methods by Transient Induced Currents.

Suppose a quantity Q of electricity passes instantaneously through a ballistic galvanometer of which the constant is G, and at which the horizontal component of the earth's magnetic field is H, the period of a complete oscillation of the needle being T. Let θ be the deflexion produced by Q. Then we have

$$Q = \frac{H}{G} \cdot \frac{T}{\pi} \sin \frac{\theta}{2}.$$

The transient current in the galvanometer may be produced in two ways.

(i) A large coil is set so as to be capable of rotating about a vertical axis in a place where the horizontal component of the earth's field is H'. The ends of this coil are connected to the galvanometer. Let the magnetic moment of the coil per unit current in it be S. This may be found from its dimensions by taking the sum of the areas of its turns, they being all in one plane. Let R be the resistance of the circuit formed of the coil and galvanometer.

The coil is set first at right angles to the magnetic meridian, and then rapidly rotated through 180°. The quantity of electricity which passes is

$$\frac{2SH'}{R}.$$

Thus we have, θ being the galvanometer deflexion,

$$R = 2SG\frac{H'}{H} \frac{\pi}{T \sin \frac{\theta}{2}}.$$

The ratio H'/H may be found by oscillating the same magnet at the centres of the galvanometer and the rotating coil.

(ii) Two coils are taken whose coefficient of mutual induction M can be calculated. One of these is connected to the galvanometer, forming a circuit of resistance R. In the other a current I is established. If this current is suddenly reversed the quantity of electricity which passes through the galvanometer is

$$\frac{2MI}{R}.$$

I is measured by means of a tangent galvanometer, whose constant is G', at a place where the horizontal component of the earth's field is H'. Let a be the deflexion produced by I in this galvanometer. Then we have

$$I = \frac{H'}{G'} \tan a.$$

From these equations we get

$$R = 2M \frac{H'}{H} \frac{G}{G'} \frac{\tan a}{T \sin \frac{\theta}{2}}.$$

The ratio H'/H may be found as before. Or the reduction factors $H/G, H'/G'$ of the two instruments may be completely got rid of by passing the same current through them; the ballistic galvanometer, since it is a very sensitive instrument, being shunted with a known shunt; and observing the deflexions produced.

The first of these two methods is due to Weber. The second was first suggested by Kirchhoff, who was the first to determine in absolute measure the resistance of a conductor. His experiment, however, was arranged in a different manner; and the theory of it is much less simple. He used the same galvanometer to measure the permanent and the induced currents. The above method, and various slight modifications of it, have been employed by several experimenters with good results.

5. (3) Lorenz's Method by Continuous Induced Current.

A conducting disc is made to rotate about its axis in a magnetic field. If the ends of a conductor are placed in contact with the centre and circumference of the disc an E. M. F. will be induced in the circuit so formed. The magnetic field is produced by a coil coaxal with the disc carrying a current. Thus the magnetic forces of the field form a system of revolution about the axis of the coil. And the quantity of magnetic induction cut across by the moving disc, considered as part of the circuit, in a single revolution, will be equal to the quantity enclosed by the rim of the disc.

Now let M be the coefficient of mutual induction of the coil and a circuit coinciding with the rim of the disc. And let I be the current carried by the coil. Let T be the time of a revolution of the disc. The E. M. F. induced by the rotation of the disc is

$$\frac{MI}{T}.$$

Now the centre and the circumference of the disc are connected with the ends of a portion of resistance R of a circuit in which a current I' is flowing. And matters are so arranged that no current flows in the disc or its connexions with R. Thus the E. M. F. induced by the motion of the disc just balances the E. M. F. RI'. And we have

$$R = \frac{MI}{TI'}.$$

The current I of the coil is made the same as that in the resistance R, and so we have

$$R = \frac{M}{T}.$$

6. (4) British Association Method by means of a Revolving Coil.

A closed circular coil is made to revolve uniformly about a vertical axis. The currents induced in this by the earth's field act on a small magnet suspended at its centre by means of

a silk fibre. Under the action of these currents the magnet would not, strictly speaking, have a position of equilibrium; but by making it of large moment of inertia and making the coil revolve rapidly it settles down, approximately, into a definite position. Its deflexion from the magnetic meridian is read by means of a beam of light reflected on a scale.

Let G be the galvanometric constant of the coil, or the magnetic intensity at its centre per unit current;

S the magnetic moment of the coil per unit current;

L its coefficient of self-induction;

R its resistance in ohms;

ω its angular velocity;

i the current in the coil at any instant;

H the horizontal component of the earth's magnetism;

M the magnetic moment of the magnet;

A its moment of inertia;

θ the angle between the magnetic meridian and the plane of the coil at any instant;

ϕ the angle between the magnetic meridian and the axis of the magnet;

α the value of ϕ when there is no torsion in the thread;

$MH\tau$ the coefficient of torsion of the suspension fibre.

The entire E. M. F. induced in the coil in consequence of its motion with respect to the earth and the needle is

$$HS\frac{d}{dt}\sin\theta + GM\frac{d}{dt}\sin(\theta-\phi)$$

$$= HS\omega\cos\theta + GM\omega\cos(\theta-\phi),$$

because $\dfrac{d\theta}{dt} = \omega$, and ultimately ϕ is constant.

Thus we have

$$Ri + L\frac{di}{dt} = HS\omega\cos\theta + GM\omega\cos(\theta-\phi).$$

Multiplying by $e^{\frac{Rt}{L}}/L$, and integrating, and writing $\theta = \omega t$, and dividing by $e^{\frac{Rt}{L}}$, we get

$$i = \frac{HS\omega}{R^2 + L^2\omega^2}(R\cos\omega t + L\omega\sin\omega t)$$

$$+ \frac{GM\omega}{R^2+L^2\omega^2}\Big\{R\cos(\omega t - \phi) + L\omega\sin(\omega t-\phi)\Big\} + Ce^{-\frac{Rt}{L}}.$$

The last term disappears after a short time.

From the motion of the needle we have

$$A\frac{d^2\phi}{dt^2} = MGi\cos(\theta-\phi) - MH\{\sin\phi + \tau(\phi-a)\}.$$

In the position of equilibrium of the needle $\dfrac{d^2\phi}{dt^2} = 0$. Thus we have $\quad Gi\cos(\theta-\phi) = H\{\sin\phi+\tau(\phi-a)\}$.

Substituting in the above equation, the terms in θ, or ωt must vanish; for the final value of ϕ does not depend on θ. Thus leaving these terms out we get

$$\frac{\omega\, G}{R^2+L^2\omega^2}\Big\{HS(R\cos\phi + L\,\omega\sin\phi) + GMR\Big\}$$
$$= 2H\{\sin\phi+\tau(\phi-a)\}.$$

From this equation the value of R can be deduced.

It is not necessary to know the value of H. The value of M/H can be determined by using the suspended magnet to deflect the needle of a magnetometer.

This method was suggested by Sir William Thomson to the Committee of the British Association on Electrical Standards, and used by them in 1863.

CHAPTER X.

Dimensions.

1. By the dimensions of a physical quantity we mean the quantities, and powers of quantities, involved in the measurement of it. Thus a velocity, since it is distance per unit of time, and is measured by a distance divided by a time, is of the dimensions of a length and the reciprocal of a time: an acceleration, since it is increase of velocity per unit of time, is of the dimensions of a velocity and the reciprocal of a time; or of a length and the inverse square of a time. Now in mechanics we know that **length, mass,** and **time** are the fundamental quantities the units of which we choose as fundamental units; and the units of all other quantities are derived from these. These three fundamental units are sufficient for the statement of the measure of any quantity that can occur in mechanics. For instance, we may say that a force is one which acting on the unit of mass for unit of time will generate in it a velocity of so many units of length per unit of time. Thus the dimensions of all quantities that occur in mechanics can be expressed in terms of those of length, mass, and time.

2. The units of all quantities that occur in Electricity and Magnetism are defined with reference to their mechanical actions, and the dimensions of all such quantities can be expressed in terms of the fundamental units of length, mass, and time.

The fundamental units of length, mass, and time are denoted by the symbols $[L]$, $[M]$, and $[T]$, and any derived unit is denoted by a similar symbol in square brackets. Thus $[LT^{-1}]$, $[LT^{-2}]$ for the units of velocity and acceleration.

3. The expression of any physical quantity may be considered to consist of two factors, one being the unit in terms of which it is measured, and the other the number of times it contains the unit. If the magnitude of the unit is altered in any manner, the numerical factor in the expression of a given quantity will be altered in the inverse ratio. Now the magnitude of the unit will depend on those of the fundamental units chosen. If we know in what manner it depends upon them, that is to say, if we know its dimensions, we can easily find how it will be changed if we change the fundamental units. For suppose that a derived unit $[N]$ is $[L^p M^q T^r]$. Then by taking new fundamental units $[L']$, $[M']$, $[T']$, the corresponding derived unit $[N']$ is $[L'^p M'^q T'^r]$. And the ratio of the new to the old unit is

$$\left(\frac{L'}{L}\right)^p \left(\frac{M'}{M}\right)^q \left(\frac{T'}{T}\right)^r.$$

4. Derived Mechanical Units. The following are the dimensions of the principal derived units that occur in mechanics.

Velocity. A velocity being distance passed over per unit of time, the dimensions of the unit of velocity are $[LT^{-1}]$.

Acceleration. This, being increase of velocity per unit of time, has for dimensions $[LT^{-2}]$.

Force. This, being measured by the product of a mass and an acceleration, has for dimensions $[LMT^{-2}]$.

Work. This, being measured by the product of a force and a distance, has for dimensions $[L^2MT^{-2}]$.

5. The dimensions of the unit of any quantity will depend upon the action with reference to which the unit has been defined. So that if we can find two entirely different actions of the same quantity with reference to which to fix its unit and to measure it, the unit will, as a rule, be of different dimensions in the two cases, and the measure of the quantity will vary when we vary the fundamental units in different manners according to the ways we have chosen of selecting the unit. As an example of different ways of selecting a unit for the same quantity we

may refer to the definitions already given of the unit of electricity according to its electrostatic and electromagnetic actions respectively; the unit in the first case being defined with respect to the action of a charge of electricity on another charge, and that in the second case with respect to the action on a magnetic pole of a quantity of electricity flowing along a conductor.

6. There are two systems of units of all the quantities that occur in electricity and magnetism, which are obtained respectively by starting with electrostatic and magnetic actions, and are called the electrostatic and electromagnetic systems. We shall determine the dimensions of the units in these two systems. We shall denote by a small letter the measure of any quantity in the electrostatic system, and by the same letter in square brackets the corresponding unit; and by the corresponding capital letter, and this letter in square brackets, the measure of the same quantity in the electro-magnetic system, and its unit.

7. ELECTROSTATIC SYSTEM.

Quantity of Electricity. A quantity q of electricity acts on an equal quantity at a distance d with a force f, where $f = q^2/d^2$.

$$\text{Thus} \qquad q = d\sqrt{f}.$$

And since d is of dimensions $[L]$ and f of dimensions $[LMT^{-2}]$, we have

$$[q] = [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}].$$

Surface-Density of Electricity, being the quantity per unit area, we have for the unit,

$$[\sigma] = [L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}].$$

Electrical Intensity, being force per unit of quantity, we have for the unit,

$$[h] = [f]/[q] = [L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}].$$

Specific Inductive Capacity is by definition, in this system, a ratio, that is to say, it is a quantity of no dimensions in the fundamental units.

Electrostatic Potential, or E. M. F., is work per unit quantity of electricity; so that we have for the unit,

$$[e] = [L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}].$$

Capacity being quantity of electricity per unit of E. M. F., we have for the unit,

$$[c] = [q]/[e] = [L].$$

Current being quantity per unit time, we have for the unit,

$$[i] = [L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-2}].$$

Resistance being defined by Ohm's law, we have for the unit,

$$[r] = [e]/[i] = [L^{-1} T].$$

Quantity of Magnetism. A quantity q' of magnetism concentrated at the centre of a circular conductor of length l and radius d and carrying a current i is acted on by a force f due to the current, where $f = ilq'/d^2$.

Thus $$q' = fd^2/il;$$

$$[q'] = [f][L]/[i] = [L^{\frac{1}{2}} M^{\frac{1}{2}}].$$

Surface Density of Magnetism being quantity per unit area, we have for the unit,

$$[\sigma'] = [L^{-\frac{3}{2}} M^{\frac{1}{2}}].$$

Magnetic Intensity being force per unit quantity of magnetism, we have for the unit,

$$[h'] = [f]/[q'] = [L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-2}].$$

Magnetic Potential being work per unit quantity of magnetism, we have for the unit,

$$[e'] = [L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-2}].$$

Magnetic Power being product of magnetic density by thickness of shell, we have for the unit,

$$[\phi] = [L^{-\frac{1}{2}} M^{\frac{1}{2}}].$$

8. ELECTROMAGNETIC SYSTEM.

Quantity of Magnetism. A quantity Q' of magnetism acts on an equal quantity at a distance d with a force f, where

$$f = Q'^2/d^2.$$

Thus $$Q' = d\sqrt{f}.$$

And we have for the unit,

$$[Q'] = [L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}].$$

Surface Density of Magnetism being quantity of magnetism per unit area, we have for the unit,

$$[\Sigma'] = [L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1}].$$

Magnetic Intensity being force per unit of magnetism, we have for the unit,

$$[H'] = [L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1}].$$

Magnetic Potential being work per unit quantity of magnetism, we have for the unit,

$$[E'] = [L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}].$$

Magnetic Power being the product of magnetic density and thickness of shell, we have for the unit,

$$[\Phi] = [L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}].$$

Current. A quantity Q' of magnetism concentrated at the centre of a circular conductor of length l and radius d and carrying a current I is acted on by a force f due to the current, where

$$f = IlQ'/d^2.$$

Thus $$I = fd^2/lQ';$$

$$[I] = [f][L]/[Q'] = [L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}].$$

Quantity of Electricity being product of current and time, we have for the unit,

$$[Q] = [L^{\frac{1}{2}} M^{\frac{1}{2}}].$$

Surface-Density of Electricity being quantity per unit area, we have for the unit,

$$[\Sigma] = [L^{-\frac{3}{2}} M^{\frac{1}{2}}].$$

Electrical Intensity being force per unit quantity of electricity, we have for the unit,

$$[H] = [L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-2}].$$

Electrostatic Potential, or E. M. F., being work per unit quantity of electricity, we have for the unit,

$$[E] = [L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-2}].$$

Capacity being quantity per unit of E. M. F., we have for the unit,

$$[C] = [L^{-1} T^2].$$

Resistance being defined by Ohm's Law, we have for the unit,

$$[R] = [E]/[I] = [L T^{-1}].$$

Specific Inductive Capacity. If Σ is the surface-density of electricity at a surface in a medium of specific inductive capacity K, and V is the potential at the surface, and n a distance measured along the normal to the surface, passing into the medium, we have

$$4 \pi \Sigma = -K \frac{dV}{dn}.$$

Thus we have for the unit of specific inductive capacity,

$$[K] = [\Sigma][L]/[E] = [L^{-2} T^2].$$

9. These are the dimensions of the units of the principal quantities occurring in electricity and magnetism. It may be noticed that the quotient obtained by dividing the unit of any quantity in either system by that of the same quantity in the other system is always some power direct or inverse of the unit of velocity. Thus, for instance, taking the units of quantity of electricity, we have

$$[q]/[Q] = [LT^{-1}].$$

From this it follows that the number of electrostatic units in an electromagnetic unit is the numerical measure of a certain velocity the absolute value of which does not depend upon the magnitudes of the fundamental units. For if these units be changed in any manner this number will vary inversely as the unit of velocity, so that the product of it and the unit of velocity, that is the velocity of which it is the numerical measure, will remain constant.

Let us denote this number by v. We can see in just the same way that the number of electrostatic units of any quantity contained in an electromagnetic unit of the same quantity is some simple power of v.

We may also see in the following way the relations that exist between the units of the principal quantities in the two systems. Let i, I be the numerical measures, in the two systems, of an electric current, with similar meanings for the other letters. We have for W, the measure in joules of a quantity of energy, the following expressions,

$$W = i^2 r t = I^2 R t,$$
$$W = eit = EIt,$$
$$W = eq = EQ,$$
$$W = e^2 c = E^2 C.$$

From these equations we get

$$\sqrt{\frac{R}{r}} = \frac{i}{I} = \frac{E}{e} = \frac{q}{Q} = \sqrt{\frac{c}{C}}.$$

Thus we have for the ratios of the units of the quantities,

$$\frac{[I]}{[i]} = \frac{[e]}{[E]} = \frac{[Q]}{[q]} = v,$$
$$\frac{[r]}{[R]} = \frac{[C]}{[c]} = v^2.$$

10. The dimensions which have been here assigned to the various electrical quantities are those usually given. They serve one purpose of dimensions; namely, to show in what ratio the unit of any quantity would be changed if the fundamental units were changed in any manner. And they further serve as a check on a formula or equation by showing whether all its terms have the same dimensions in L, M, and T.

The dimensions of a physical quantity in terms of the fundamental mechanical units, ought, however, if possible, that is to say, when our knowledge is sufficient, to indicate the mechanical

nature of the quantity. As for instance, in mechanics, the dimensions of force $[L\,M\,T^{-1}]$, show us that force involves the ideas of a mass moving with acceleration, and shows how the magnitude of the mass and that of the acceleration enter into the measure of the force.

Now in the case, for example, of the electrostatic unit of quantity, the *result* of the mechanical actions of two quantities of electricity is involved in the definition, and so comes into the dimensions; but what the ultimate mechanical nature of an electrical charge is, is necessarily, on account of our ignorance of it, in no way indicated in the dimensions.

That the natures of the quantities in the two systems are not indicated in the dimensions would be obvious from the mere fact that the dimensions are different in the two systems. In finding the dimensions in either case we start by contemplating the *results* of two sets of actions; the actions chosen in the two cases being different, thus leading to results, from this point of view, inconsistent with each other. If we knew enough about electricity and magnetism to start in the two cases with the quantities themselves, we should doubtless get the same dimensions for any quantity in the two systems, if two systems were then necessary. In either case the actions, whatever they are, take place in the surrounding medium, and depend on that medium, that is on the quantity K or μ.

Professor Rücker has suggested that these quantities should be regarded, not as mere ratios, but as quantities having dimensions, although, at present, unknown dimensions, in L, M, and T. These he has called *secondary fundamental units*, and has proposed their introduction into the dimensions of electrical and magnetic quantities, showing how in the ordinary expression of these dimensions the quantities K and μ are *suppressed*.

In the electrostatic system, then, for unit of quantity we must write our relation $f = q^2 / d^2 K$.

Thus $$[q] = [L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1} K^{\frac{1}{2}}].$$

In the electromagnetic system, for the dimensions of unit quantity of magnetism we have, in the same way,

$$[Q'] = [L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1} \mu^{\frac{1}{2}}].$$

For unit of current

$$f = I l Q' / d^2;$$

$$\therefore\ [I] = [L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}].$$

For unit of electricity

$$[Q] = [L^{\frac{1}{2}} M^{\frac{1}{2}} \mu^{-\frac{1}{2}}].$$

In the same way the dimensions of all the other quantities in the two systems, with K and μ introduced, may easily be found.

By comparing the dimensions of any quantity in the two systems, a necessary relation between those of K and μ is seen. Thus considering that $[q]$ and $[Q]$ have the same dimensions we see that the dimensions of $K^{-\frac{1}{2}} \mu^{-\frac{1}{2}}$ are $[LT^{-1}]$.

11. Experiments have been made to determine the value of v with reference to various pairs of units; these have all given for it, supposing we take for fundamental units those of the C.G.S. system, a value of about 30×10^9. This is about the same as the measure of the velocity of light with the same units; the two numbers, the ratio v, and the measure of the velocity of light agreeing to about the same degree of accuracy as the various determinations of either of them agree among themselves. If then we suppose these two quantities to be the same, we see that whatever be the fundamental units chosen, the number of electrostatic units of electricity in an electromagnetic unit is a number expressing a velocity which is always the velocity of light. If our fundamental units were so chosen that the velocity of light should be unit of velocity, then the units of any quantity in the electrostatic and electromagnetic systems would be equal.

We may obtain a physical conception of the velocity v, as it relates to the units of electricity, in the following manner. Suppose we have two parallel plane surfaces electrified to surface

densities σ, σ'. The force of repulsion on either of these per unit area is $2\pi\sigma\sigma'$.

Now let us suppose that an electrified body in motion is equivalent to an electric current. And let these two surfaces move in a direction parallel to themselves with velocity v. This is equivalent to two parallel current-sheets; and there will in consequence be attraction between the planes. If u, u' are the surface-densities of the current in the two sheets, or the quantities of electricity flowing per second across unit length drawn at right angles to the direction of flow, it may be shown that the consequent attraction on either plane per unit area is $2\pi u u'$; where u and u' are measured in electromagnetic units. Now let n be the number of electrostatic units in an electromagnetic unit. The surface-densities of current due to the motion of the two planes measured in electrostatic units are $\sigma v, \sigma' v$. Thus we have

$$u = \frac{\sigma v}{n}, u' = \frac{\sigma' v}{n}.$$

Now let us suppose that the velocity v is such that the electrostatic repulsion is just balanced by the electromagnetic attraction. Then we must have

$$2\pi\sigma\sigma' = 2\pi\frac{\sigma\sigma' v^2}{n^2}.$$

Therefore $$v = n.$$

This gives the velocity with which the surfaces must move, and shows that the number expressing it is equal to the number of electrostatic units contained in an electromagnetic unit.

It has not been shown whether a moving electrified body does produce the same electromagnetic effect as that here supposed, but Rowland has shown that such a body does produce some magnetic effect, of the same order of quantities as we have supposed.

12. We have seen that the quantity $1/\sqrt{K\mu}$ is of the dimensions of a velocity. Let us consider what particular velocity it represents.

Suppose the measures *on any arbitrary system whatever* of the specific inductive capacity and permeability of a given medium to be K and μ. And suppose the measures, on the same system, of the electrostatic and magnetic units of quantity, the unit of magnetism, and the unit of current to be q, Q, Q', I; these being the units with regard to the given medium and some given system of fundamental units. Take the four equations connecting these quantities mechanically,

$$f = q^2/d^2 . K,$$
$$f = Q'^2 / d^2 . \mu,$$
$$f = I . l . Q' / d^2,$$
$$Q = I . t.$$

Now by making $d = 1$ in the first of these, so that $f = 1$, and similarly with the others, we get

$$K = q^2,$$
$$\mu = Q'^2,$$
$$IQ' = 1,$$
$$I = Q.$$

From which

$$K\mu = q^2 Q'^2,$$

$$\frac{1}{\sqrt{K\mu}} = \frac{Q}{q}.$$

That is in any arbitrary system whatever the number which measures $1/\sqrt{K\mu}$ is the number expressing the ratio of the electromagnetic to the electrostatic unit on the same system and with reference to the medium. Thus $1/\sqrt{K\mu}$ denotes the velocity v above mentioned. (Arts. 9, 11.)

In the case of air this quantity has been shown by experiment to be equal to the velocity of light. In his Electromagnetic Theory of Light, Maxwell supposes that for any medium $1/\sqrt{K\mu}$ would be the velocity of light in that medium, being the velocity of propagation of electromagnetic disturbances, and light being a manifestation of these disturbances.

Thus, taking symbols to refer to two different media,

$$\sqrt{\frac{K_2 \mu_2}{K_1 \mu_1}} = \frac{v_1}{v_2},$$

and is thus the optical index of refraction from the first to the second mediums.

Taking K and μ, as is generally done, to simply mean the ratios of the corresponding quantities in a given medium to those in air, we see that $\sqrt{K\mu}$ denotes the index of refraction of that medium with respect to air.

13. Weber and Kohlrausch's determination of v. Weber and Kohlrausch were the first to make a determination of the value of v. They determined it as the ratio of the electromagnetic unit of electricity to the electrostatic unit, or the ratio of the numbers expressing the value of the same quantity of electricity in electrostatic and electromagnetic units.

The following is the principle of the method. A Leyden jar of known capacity was charged, and the electrostatic measure of the charge was determined. The jar was then discharged through a ballistic galvanometer to determine the electromagnetic measure of the charge. The ratio of the first of these measures to the second is the required value of v.

The capacity of the jar was determined by comparing it with that of an isolated sphere, the capacity of which in electrostatic measure is equal to its radius. This was done by putting the jar in contact with the sphere several times, and putting the sphere to earth after each contact, and comparing the potential of the jar, before these contacts, with that which it has after they have been made. The potential of the jar is diminished at each contact in the ratio of its capacity to the sum of the capacities of the jar and sphere.

The potential of the jar could now be determined by means of an electrometer of known constants; and so the charge could be found as the product of capacity and potential. But in the original experiments a torsion-balance, of known constants, and having the fixed and moving balls equal, was used,

With the fixed ball, of known size, a known fraction of the sphere's charge was taken from it, and thus a known fraction of the charge of the jar. Half this charge was imparted by contact to the moving ball, and by the deflexion, and the known constants of the instrument, the value of the charge, and thus of the charge of the jar, was determined.

14. Sir Wm. Thomson's determination of v. Thomson determined v as the ratio of the electrostatic to the electromagnetic unit of E. M. F.

A current was sent through a coil whose resistance R in electromagnetic measure was known. The value of the current I was determined by means of an electrodynamometer, so that it was not necessary to determine the value of H. Thus the electromagnetic measure E of the E. M. F. of the ends of the coil was known, being equal to RI.

The electrostatic measure e of the same E. M. F. was determined by connecting the extremities of the resistance R with an absolute electrometer. Thus we get for the required value

$$v = \frac{E}{e}.$$

In this method of determining v it is necessary to know the value of R in absolute measure, as this value is a factor in the result, and if it has been measured in terms of any given unit, for instance the B.A. unit, we must know how many C.G.S. units of resistance this contains. The other factor of the result, it may be seen, is merely a numerical quantity, and of no dimensions in the units; for instance the values of I and of e both involve the square root of a force. This method may then be considered as one for determining v as a resistance.

15. Determination of v by means of the measures of a capacity.

The value of v is the square root of the ratio of the electromagnetic to the electrostatic unit of capacity.

Suppose we have a condenser whose electrostatic capacity can be readily calculated from its dimensions; such as one formed of two parallel circular plates with a guard-ring.

The electromagnetic measure of the capacity may be determined in the following way. A current is passed through a known resistance R, and its value determined by the deflexion α produced in a galvanometer, from the formula

$$I = \frac{H}{G} \tan \alpha.$$

The ends of the resistance R are connected to the electrodes of the condenser, thus charging it to E.M.F. RI. If then C is the capacity of the condenser it is charged with a quantity of electricity,

$$CR \frac{H}{G} \tan \alpha.$$

If the condenser is now discharged through the galvanometer, the period of a complete vibration of which is T, producing a swing δ, we have the quantity discharged

$$\frac{T}{\pi} \frac{H}{G} \sin \frac{\delta}{2}.$$

Thus we get

$$C = \frac{T}{\pi} \frac{1}{R} \frac{\sin \frac{\delta}{2}}{\tan \alpha}.$$

If c is the electrostatic measure of the condenser, as determined by measurement, we have

$$v^2 = \frac{\pi R c}{T} \frac{\tan \alpha}{\sin \frac{\delta}{2}}.$$

This method has been used by Professors Ayrton and Perry. To make it workable, matters were arranged so that the entire current I was not passed through the galvanometer, which, as it was used to measure the discharge, was a very sensitive one. But the galvanometer, of resistance g, was shunted with a shunt s, and a very great resistance ρ added to it. The terminals of the resistance thus formed were then connected with those of a known fraction m of the entire resistance R. This practically

does not alter the value of the current in mR, and so we observe the current produced by the E.M.F. mE in the resistance $\rho+\dfrac{gs}{g+s}$. If α' is the deflexion thus produced, we have

$$mE=\frac{\rho(g+s)+gs}{s}\frac{H}{G}\tan\alpha',$$

since the current in the resistance $\rho+gs/(g+s)$ is

$$\frac{g+s}{s}\cdot\frac{H}{G}\tan\alpha'.$$

And the charge is CE; thus we get

$$C=\frac{ms}{\rho(g+s)+gs}\frac{T}{\pi}\frac{\sin\frac{\delta}{2}}{\tan\alpha'}.$$

CHAPTER XI.

EXAMPLES OF ELECTRO-MAGNETIC MEASUREMENTS.

1. To compare a Coefficient of Mutual-Induction and a Resistance. The primary coil is joined up in circuit with a constant battery so that a current I flows in it. The secondary is connected with a ballistic galvanometer. Let M be the coefficient of mutual-induction of the coils; R the resistance of the circuit composed of the secondary coil and galvanometer. The current in the primary coil is suddenly reversed, and the swing of the galvanometer, a, read. The entire quantity of electricity that has passed through the galvanometer is $2MI/R$.

Thus $$\frac{2MI}{R} = \frac{T}{\pi} \cdot \frac{H}{G} \sin\frac{a}{2}\left(1 + \frac{\lambda}{2}\right).$$

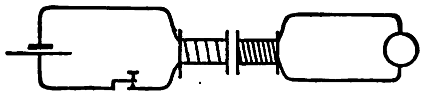
Next the galvanometer is shunted to a portion of the primary circuit of small resistance, S; and the resistance, K, of the galvanometer circuit is made very large. Thus the entire current in the primary circuit may be supposed to be unaltered. And the current through the galvanometer will be $IS/(S+K)$. Let β be the deflexion obtained.

Thus $$\frac{IS}{S+K} = \frac{H}{G}\tan\beta.$$

$$\therefore \quad \frac{2M}{R} = \frac{S}{S+K}\frac{T}{\pi} \cdot \frac{\sin\frac{a}{2}}{\tan\beta}\left(1 + \frac{\lambda}{2}\right).$$

This method may be used to determine M if we know the value of R and the ratio of S and K. Or it may be used to determine the absolute value of a resistance R, if the value of M is known by calculation and the relative values of the other resistances can be found.

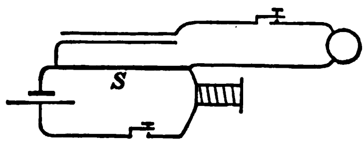
2. To compare a Coefficient of Mutual-Induction and a Capacity. Join up a constant battery in circuit with the primary coil, and a ballistic galvanometer with the secondary coil, as in the first figure. Let M be the coefficient of mutual induction of the coils;



R the entire resistance of the secondary circuit; I the current in the primary circuit. Observe the throw, a, of the galvanometer on making the primary circuit.

Then $$\frac{MI}{R} = \frac{T}{\pi} \cdot \frac{H}{G} \cdot \sin\frac{a}{2}\left(1 + \frac{\lambda}{2}\right).$$

Next charge the condenser of capacity C, at the extremities of a portion of known resistance S of the primary circuit, through the galvanometer, while the current I is flowing through the

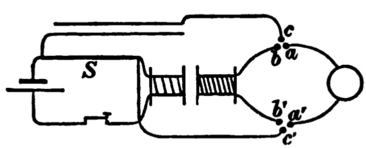


primary circuit. Observe the throw β of the galvanometer.

Then $$CSI = \frac{T}{\pi} \cdot \frac{H}{G} \cdot \sin\frac{\beta}{2}\left(1 + \frac{\lambda}{2}\right).$$

Therefore $$\frac{M}{CRS} = \frac{\sin\frac{a}{2}}{\sin\frac{\beta}{2}}.$$

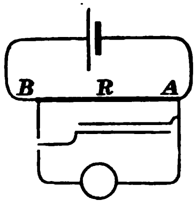
The change from the first to the second arrangement may be made in practice by means of two three-way pieces abc, $a'b'c'$. a and a' are connected to b and b' to put the galvanometer in



circuit with the secondary coil; and with c and c' to charge the condenser.

3. Measure of Capacity of a Condenser in terms of

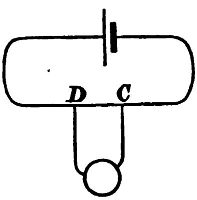
a Resistance, with Ballistic Galvanometer. A constant battery is joined up in circuit with a very large resistance. Let AB be a portion of this resistance of known value R. The condenser is arranged as in the first figure, so that it may be charged to the P. D. of A and B and discharged through the galvanometer.



Let I be the current through AB, C the capacity of the condenser, α the swing of the galvanometer when the condenser is discharged through it, other letters having their usual meaning. Then, since CIR is the quantity of electricity that passes through the galvanometer,

$$CIR = \frac{T}{\pi} \cdot \frac{H}{G} \sin \frac{\alpha}{2}\left(1 + \frac{\lambda}{2}\right). \dots \qquad (1)$$

Next the galvanometer is shunted on to a portion CD of the main circuit of small resistance S, and the resistance, K, of the galvanometer circuit is made very large.



Let β be the steady deflexion obtained in the galvanometer. The current in the main circuit may be supposed still to have the value I, since the resistance of the circuit has been very slightly diminished. The current in the galvanometer is now $IS/(S+K)$.

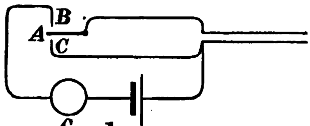
Thus $$\frac{IS}{S+K} = \frac{H}{G} \cdot \tan \beta. \dots \qquad (2)$$

From equations (1) and (2) we get

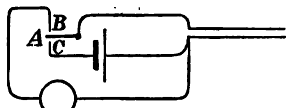
$$C = \frac{S}{S+K} \cdot \frac{1}{R} \cdot \frac{T}{\pi} \cdot \frac{\sin \frac{\alpha}{2}}{\tan \beta}\left(1 + \frac{\lambda}{2}\right).$$

To determine C in this way we must know the value of R accurately. We only require to know the ratio of the resistances S and K.

4. Measure of Capacity of Condenser by means of Intermittent Currents. Suppose the condenser arranged with a galvanometer G, and a battery of E. M. F. E, and a piece A moving between the two contacts B and C, either as in fig. 1, so that when A touches B the condenser is charged through the galvanometer, and when A touches C it is discharged; or as in fig. 2, so that when A touches C the condenser is charged, and when A touches B it is discharged through the galvanometer.



G 1.



G 2.

Let C be the capacity of the condenser.

Then at every complete to-and-fro vibration of A a quantity EC of electricity passes through the galvanometer.

Now let A vibrate so quickly as to keep up a continuous steady deflexion of the galvanometer. Let N be the number of vibrations made by A per second. The mean current through the galvanometer is NEC.

Now let the same battery be used to drive a steady current through the galvanometer, and let the resistance of the entire circuit be adjusted to such a value R that the same deflexion of the galvanometer is obtained as before. The current is now equal to E/R.

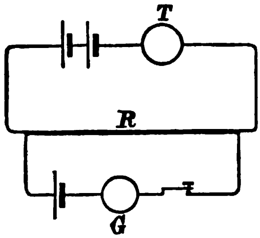
Thus we have $$NEC = \frac{E}{R},$$

$$\therefore \quad C = \frac{1}{NR}.$$

For the vibrating piece A a prong of a tuning-fork of known vibration frequency may be used.

5. To measure the E. M. F. of a Battery in terms of a given Resistance and Current. Join up a constant battery

in circuit with a known resistance R, and a tangent galvanometer T or some apparatus for measuring the current. Join the poles of the given battery through a sensitive galvanometer G to the extremities of the resistance R, so that its positive pole and the positive pole of the other or auxiliary battery are connected to the same extremity of R. Arrange the resistance R and the current through it so that when the circuit through the galvanometer G is completed it gives no deflexion.

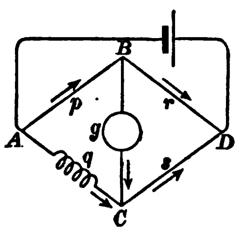


Let I be the current in R as given by T; and E the E. M. F. of the given cell. Then applying Kirchhoff's second Law [see Chap. II, § 7] to the circuit of the given cell, the galvanometer G, and R, we get

$$E = IR.$$

It is clear that the auxiliary battery must have a larger E. M. F. than that which we are to measure.

6. To measure a Coefficient of Self-Induction. The coil of resistance q whose coefficient is to be measured is joined up in a Wheatstone's bridge, and the balance obtained for steady currents, so that when the battery circuit is first made and then the galvanometer circuit there is no deflexion of the galvanometer. Let L be the coefficient of self-induction of the coil, q, L' that of the galvanometer. The arms p, r, s should have no self-induction.



The throw of the galvanometer needle is obtained when the galvanometer circuit is first made, and then the battery circuit. Let this be a.

Let the currents at any instant in the arms p, q, &c. be denoted by P, Q, &c. Let their final values in the steady state be denoted by P', Q', &c.

Then we have from the expressions for the P. D. between B and C,

$$qQ-pP+L\frac{dQ}{dt}=gG+L'\frac{dG}{dt}.... \qquad (1)$$

And $$rR-sS=gG+L'\frac{dG}{dt};$$

that is, $$rP-sQ=(g+r+s)G+L'\frac{dG}{dt}.... \qquad (2)$$

Multiply (1) by s and (2) by q, and add; and we get, since $ps=qr$,

$$sL\frac{dQ}{dt}=\left\{g(s+q)+q(r+s)\right\}G+(s+q)L'\frac{dG}{dt}.$$

Integrate this with respect to the time from the starting of the currents to the time that their steady values are attained. Let E be the entire quantity of electricity that passes through the galvanometer, $\int G\,dt$. Also since the initial and final values of the current in the galvanometer are both zero the coefficient of L' disappears. Thus we get

$$sLQ'=\{g(s+q)+q(r+s)\}E.$$

Now upset the balance slightly by putting in a very small additional resistance δp in the arm p. Let G be now the steady current produced in the galvanometer, and β the corresponding deflexion. The steady current in p may be taken to be P', the same as before, on account of the smallness of δp.

We have now

$$P's\delta p=\{g(s+q)+q(r+s)\}G. \quad \text{[See Chap. IV, § 8.]}$$

Thus $$\frac{LQ'}{P'\delta p}=\frac{E}{G};$$

$$\frac{Lr}{s\delta p}=\frac{\frac{H}{\Gamma}\frac{T}{\pi}\left(1+\frac{\lambda}{2}\right)\sin\frac{a}{2}}{\frac{H}{\Gamma}\cdot\tan\beta},$$

where Γ is the galvanometer constant, T the period of a complete vibration, and λ the logarithmic decrement.

Thus $$L=\frac{s\delta p}{r}\cdot\frac{T}{\pi}\frac{\sin\frac{a}{2}}{\tan\beta}\left(1+\frac{\lambda}{2}\right).$$

7. Lord Rayleigh's Modification.—Lord Rayleigh has modified this method, which was given by Maxwell, as follows. He found that to obtain a suitable value for β, δp could not be made so small that the current in p may be supposed still to have the value P'. Suppose the current in p, when the resistance δp is added to p, is P''. Then we have, putting P'' instead of P',

$$\frac{LQ'}{P''\delta p} = \frac{E}{G}.$$

Or, as we may write it,

$$\frac{LQ'}{P'\delta p}\frac{P'}{P''} = \frac{E}{G}.$$

Now the ratio P'/P'' may be found very approximately as follows. The battery resistance being very small we may suppose the P.D. between A and D to be constant; and the galvanometer resistance being very great we have, approximately, the same current flowing through p and r. Thus the current along ABD is inversely proportional to the resistance of ABD. And we have

$$\frac{P'}{P''} = \frac{p+\delta p+r}{p+r};$$

$$\therefore\ L = \frac{p+r}{p+\delta p+r}\cdot\frac{s\delta p}{r}\frac{T}{2\pi}\frac{a}{\beta}\left(1+\frac{\lambda}{2}\right).$$

CHAPTER XII.

Dynamos and Motors.

1. We have seen that

(1) When a conductor is moved in a magnetic field there is an E. M. F. induced in it;

(2) When a conductor situated in a magnetic field carries a current in any direction but that of the lines of induction of the field there is a tendency for it to move in the field.

The first of these principles is used in making machines for converting mechanical into electrical energy.

The second is used in making machines for converting electrical into mechanical energy.

2. The simplest machine for producing an electric current by means of mechanical motion is Faraday's disc. A disc of copper is mounted in a vertical position on a copper axis. The axis and the rim are connected by means of a wire, one end of the wire pressing on the axis, and the other dipping into mercury, into which the rim dips too. The disc is placed in a magnetic field so that the lines of induction of the field run across it. When it is rotated, the lines of induction of the field being continually intersected by it, an E. M. F. will be induced, and a current will flow in the circuit.

Suppose the magnetic field to be a uniform one; and its intensity resolved at right angles to the disc to be H. Let r be the radius of the disc; and ω its angular velocity of rotation.

Then the same E. M. F. will be induced along each radius of the disc. The quantity of magnetic induction passed across by

each radius of the disc per second will be the expression for the induced E. M. F., which is therefore

$$\frac{Hr^2\omega}{2}.$$

8. The same machine may be used for converting electrical into mechanical energy. If a current is passed from the axis to the circumference, and if the disc is free to rotate, it will do so.

The rotation of the disc produced by a current in a given direction will be opposite to the rotation which would produce a current, by induction, in the same direction. The motion will therefore induce an E. M. F. opposed to the given current, which will thus be diminished. This E. M. F. is called the **back E. M. F.**

Consider the mechanical work done in turning the disc. Let I denote the intensity of the current, the other symbols having the same meanings as before.

Then since the quantity of induction inclosed per second by any radius of the disc is $Hr^2\omega/2$, the work done per second is

$$\frac{IHr^2\omega}{2}.$$

Thus the moment of the forces acting on the disc, about its axis, is

$$\frac{IHr^2}{2}.$$

Let E denote the E. M. F. of the battery which drives the current, E' the back E.M.F. due to the rotation of the disc, and which is equal to $Hr^2\omega/2$, R the resistance of the circuit.

Then the work done per second by the battery is EI.

Of this RI^2 is used in heating the circuit.

The part expended in mechanical work is $E'I$.

Thus we have $$EI = RI^2 + E'I.$$

DEF. **THE EFFICIENCY of an arrangement used for converting electrical into mechanical energy is the ratio of the mechanical energy developed per second to the electrical energy consumed per second.**

Thus the efficiency of the arrangement above described is E'/E.

4. A machine used for obtaining electric currents from the energy of mechanical motion is called a **Dynamo.** Faraday's rotating disc used to generate a current is a simple example of a dynamo. We shall now consider the principles on which a dynamo is usually constructed for practical work.

We have seen that if a closed coil is rotated in a magnetic field, about an axis in its plane at right angles to the lines of force of the field, and has its ends always connected to those of a wire, an alternating current is produced in the circuit formed of the coil and the wire.

Now let A and B denote the ends of the coil, and C and D those of the wire: and instead of having A constantly connected to C and B to D suppose that these connexions only last for half a turn, finishing with a position in which the coil encloses the maximum quantity of magnetic induction; that is, all the time that E.M.F. is being induced in the coil in one direction. Let A then become connected to D and B to C for the next half-turn, changing to the original connexions at the end of it; and so on. We thus get a series of alternating currents in the coil, but a current in the wire in a constant direction, but varying in intensity.

This changing of the connexions is effected by means of a piece of apparatus called a **commutator,** which consists of a cylindrical copper ring placed on the axis of the coil, and cut through in two places diametrically opposite to each other. The ends of the coil are connected to these two halves of the ring, which are insulated from each other. Strips of copper to which the ends of the wire are connected press on the split ring as it rotates with the coil.

The strips of copper pressing against the commutator are called **brushes.**

We thus see how a current in a wire, uniform in direction, may be obtained from the mechanical work expended in rotating a coil in a magnetic field.

In machines used for the production of electric currents in this manner, the apparatus, consisting of coil, axis, and commutator, is called an **armature.**

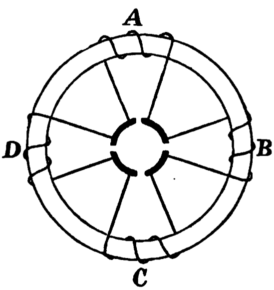
5. The next step is to obtain a current more uniform in intensity. This may be done in two different ways, in each of which an armature is used consisting of several coils, instead of one, with a commutator divided into several segments.

(1) **Open-coil Armature.** Suppose we have two identical coils, set on a common axis, at right angles to each other; and a commutator divided into four segments; the ends of each coil being connected to two opposite segments of the commutator. Let the brushes be connected by the commutator with a coil for the period of a quarter of a revolution during the time that induction in it is most powerful, and to become connected with the other coil just before the connexions with the first are broken. We thus get a current which is much more uniform in intensity than that obtained from a single coil, never falling to the value of zero.

By increasing the number of coils and commutator segments we obtain a more and more uniform current.

With an armature of this sort the brushes and commutator segments may be so made as to draw the current from more than one coil at the same time, so as to make the resistance of the armature smaller. This plan is used in the Brush machine.

(2) **Closed-coil Armature.** In this armature the coils are all in electrical connexion with each other and with the segments of the commutator, and in working the machine the induction in every coil is used at the same time.



As a type of closed-coil armatures we will describe that of the Gramme machine. The figure is a simple diagram of a Gramme armature with four coils wound on a ring made of iron wires, called the core of the armature, and used to increase the magnetic induction through the coils. The ends of each coil are connected to two adjacent segments of the commutator.

If the armature is placed in a magnetic field so that the lines of

induction of the field are in the plane of the armature, these lines will to a great extent follow the direction of the substance of the armature core, and the field will thus be far from uniform.

Now suppose in the figure the lines of induction enter the armature on the left hand side. They then pass chiefly by means of the iron of the ring through the coils and out again on the right hand side. The ring *A* is enclosing its maximum quantity of induction in one direction. Let the armature be rotating in the clock-wise direction. As it rotates *A* constantly loses magnetic induction till it comes to the position *C* when it is enclosing the maximum quantity in the other direction. As *A* ascends to the highest position again it is constantly gaining magnetic induction in the same direction as at first.

Thus we see that in a coil on the right hand side of the ring an E. M. F. is always induced in one direction, and in a coil on the left hand side an E. M. F. is always induced in the other direction. Each of these means an induction of E. M. F. from the highest to the lowest part of the commutator, or *vice versa*.

The brushes are placed at these two points of the commutator between which E. M. F. is being induced in every coil. Thus the current divides itself in two between the two parts of the armature between the brushes; and the E. M. F. induced in every coil of the armature is used at every instant.

Other closed-coil armatures, such as the Siemens, are made of cylindrical shape, called **drum** armatures, the core being made of wood, wound over with iron wire circumferentially, in old machines; and built up of iron discs placed so as to have the spindle for axis in the newer ones. The coils are wound on to the cylindrical core in rectangular shape, the wire passing along the sides and across the ends. The two ends of each coil are connected to two adjacent segments of the commutator. The generation of the current in such an armature is the same as in the Gramme ring armature.

6. Magnetic Field. The magnetic field is produced by means of strong magnets terminating in pole pieces between which the armature is rotated, the axis of the armature being at right angles

to the general direction of the lines of induction of the field. These magnets are called the **field-magnets** of the dynamo.

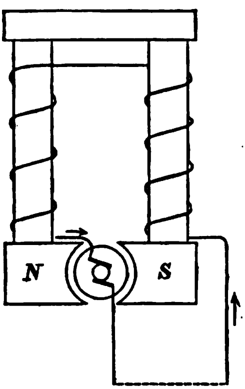
The field-magnets may be permanent, and then the dynamo is called a **magneto** machine. But they are usually electro-magnets with soft iron cores, the current being supplied by the dynamo itself.

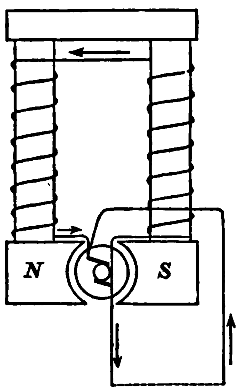
The conductors joined up to the dynamo, through which the current is to be driven, form what is called the outside circuit.

The field-magnets may be excited in two different ways.

(1) A Series Dynamo is one in which the armature, field-magnet coils, and outside circuit are in series; the entire current generated in the armature then passes through the outside circuit and the magnet coils.

(2) A Shunt Dynamo is one in which the field-magnet coils form a shunt to the outside circuit. The current generated in the armature divides itself between the outside circuit and the magnet coils.





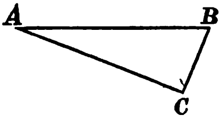
The accompanying figures show diagrams of a series and a shunt dynamo respectively.

7. Position of Brushes. The rotation of the armature in the field produces a point of maximum, and a point of minimum potential on the commutator; these two being diametri-

cally opposite each other, and the potential varying in a uniform manner from one to the other either way round the commutator, if the field-magnets and armature are well made. It is at these two points that the brushes should be placed. When this is done a brush is never connecting two segments between which there is much potential difference, and sparking at the brushes is reduced to a minimum. If the brushes are placed in any other position, each of them is in a region in which there is considerable variation of potential from one segment of the commutator to another, and thus in connecting two segments together sparking will be produced.

8. Armature Reaction.—Lead of Brushes. If no current were passing in the armature the points of maximum and minimum potential would be on that diameter of the commutator which is at right angles to the general direction of the magnetic field. But the passage of a current in the armature, by making it a magnet, influences the magnetic field, changing the direction of the lines of induction through the armature: and thus the position of the brushes must be changed, until such a position is found for them that there is no sparking; that is, they are at the points of maximum and minimum potential on the commutator. The angle through which the brushes must be turned from the diameter at right angles to the direction of the field is called the **lead** of the brushes.

Suppose an armature in a field whose lines of induction run from left to right to be rotated in the clock-wise direction. It is easily seen that the currents induced in it will make it a magnet with its S. pole above, and its N. pole below. Thus the lines of induction in it, being from S. to N. pole in the magnet, will be from above downwards. The general way in which the field will be altered can be shown by supposing the original field and that due to the magnet to be both uniform.



Draw AB to denote the original field-strength, BC that due to the magnet. Then AC will denote the resultant field-

strength to which the induced E. M. F. is due. Thus BC, which is parallel to the line of the brushes, is at right angles to AC. BAC is the angle of lead. The lead of the brushes is in the direction in which the armature rotates.

Since $AC = AB \times$ cosine of angle of lead, it is best to avoid a great lead, which always diminishes the field-strength. This is done by having few turns in the armature coils, and compensating by having powerful field-magnets.

9. **Alternate Current Dynamos.** Some machines, chiefly for electric lighting purposes, are made to give alternating currents. In these no commutator is used, but the armature terminals are connected to two rings on the axis against which the brushes press, each terminal being thus constantly in contact with a terminal of the outside circuit.

In these machines the field-magnets are generally separately excited by means of a continuous-current dynamo.

In many alternate-current dynamos the field is formed of a lot of electro-magnets placed round the circumference of a circle, giving lines of induction passing first in one direction and then in the opposite as we go round the circle. The armature consists of a lot of coils in series which move between the magnets and are so connected that the induction is always in the same direction in every coil. As the coils move from point to point of the field enclosing lines of induction, first in one direction and then in the other, a constantly alternating current is produced. The coils are generally made of a few turns of thick wire or of copper strips, to avoid the loss of energy that would arise from self-induction, if there were many turns; the field-magnets must therefore be made very strong.

10. The electrical **out-put** of a dynamo, that is, the electrical energy developed per second, is given, in watts, in the case of a continuous current machine, by the product of the E. M. F. developed in the armature and the current which passes through the armature, these being measured in volts and amperes. In the case of an alternating current machine the out-put is given by the mean value of this product.

The **Efficiency** of a dynamo is the ratio of the electrical out-put per second to the work done on it per second.

We must distinguish between two different efficiencies:

(1) The **Intrinsic** efficiency; in which the entire electrical out-put per second is considered:

(2) The **Commercial** efficiency; in which only the useful part of the electrical out-put, or the electrical energy consumed in the outside circuit, is considered. This is less than the intrinsic efficiency on account of the energy lost in the armature and other parts of the machine.

The efficiency of a given dynamo will depend on various circumstances; the speed at which it is being driven, and the current which it is sending. For a given machine it is always important to find the most advantageous conditions under which to work it.

11. Out-put of Alternate-Current Dynamo. Let R be the entire resistance, L the coefficient of self-induction of armature and outside circuit. Let the time taken for the magnetic induction to be reversed in every coil of the armature and to return to its original value be T. Suppose the E. M. F. of the armature to be given, as in the case of a rotating coil, by a sine function, as follows,

$$E = E_0 \sin \frac{2\pi t}{T}.$$

The equation of the current is

$$E = RI + L\frac{dI}{dt}.$$

Thus $$\frac{E_0}{L} \sin \frac{2\pi t}{T} = \frac{RI}{L} + \frac{dI}{dt}.$$

Multiplying by $e^{\frac{Rt}{L}}$, and integrating,

$$e^{\frac{Rt}{L}}.I = \frac{E_0}{L} \cdot \frac{e^{\frac{Rt}{L}}\left(\frac{R}{L}\sin\frac{2\pi t}{T} - \frac{2\pi}{T}\cos\frac{2\pi t}{T}\right)}{\frac{R^2}{L^2} + \frac{4\pi^2}{T^2}} + C.$$

Or, when the steady state is attained,

$$I = \frac{E_0}{L} \frac{\sin\left(\frac{2\pi t}{T} - \phi\right)}{\sqrt{\frac{R^2}{L^2} + \frac{4\pi^2}{L^2}}},$$

putting $\tan\phi = \frac{2\pi L}{RT}$.

Thus the current does not attain its maximum value at the same time as the E. M. F., but lags behind.

The out-put per second is given in watts as follows,

$$W = \frac{1}{T}\int_0^T EI\,dt$$

$$= \frac{1}{T} \cdot \frac{E_0^2}{L\sqrt{\frac{R^4}{L^2} + \frac{4\pi^2}{T^2}}} \int_0^T \sin\frac{2\pi t}{T} \sin\left(\frac{2\pi t}{T} - \phi\right) dt$$

$$= \frac{E_0^2}{2TL\sqrt{\frac{R^2}{L^2} + \frac{4\pi^2}{T^2}}} \int_0^T \left[\cos\phi - \cos\left(\frac{4\pi t}{T} - \phi\right)\right] dt$$

$$= \frac{E_0^2}{2TL\sqrt{\frac{R^2}{L^2} + \frac{4\pi^2}{T^2}}} \,.\cos\phi\,.\,T$$

$$= \frac{RE_0^2}{2L^2\left(\frac{R^2}{L^2} + \frac{4\pi^2}{T^2}\right)}$$

$$= \frac{E_0^2}{2\left(R + \frac{4\pi^2 L^2}{RT}\right)}.$$

12. Motors. When a current is driven through the armature of a dynamo; its field-magnets also being excited, if they are

electromagnets; the mechanical action on its coils will cause the armature to rotate. And, by the law of Lenz, this rotation will be in the opposite direction to that which would generate a current in the same direction as that used.

The armature of a dynamo caused to rotate in this manner may be made to do useful work. We have thus a means of converting electrical into mechanical energy. A dynamo used in this way is called an **electro-motor**, or, more simply, a **motor**.

The **Efficiency** of a motor is the ratio of the mechanical out-put per second to the electrical energy consumed by it per second.

As in the case of the dynamo, we must distinguish between two different efficiencies:

(1) The **Intrinsic** efficiency, in which we consider the whole of the work done by the motor:

(2) The **Commercial** efficiency, in which we consider only the useful work done by the motor. This is less than the intrinsic efficiency on account of the work lost in heating bearings, &c.

Suppose a current I to be passing through a motor which is a series or magneto machine, E being the E.M.F. at its terminals; let it be running at such a speed as to develop a back E.M.F. E', and let R be the electrical resistance through it from terminal to terminal. Then we have the relation

$$EI = RI^2 + E'I.$$

Now EI is the energy expended per second. This is used partly in heating the conductors, and partly in developing mechanical work. RI^2 is the measure of the heat developed per second in joules. The mechanical work done per second in joules is $E'I$.

We can easily see that $E'I$ is the work done per second, by remembering that this is the product of the current and the quantity of induction enclosed per second by the coils of the armature; and the latter is equal to the back E.M.F..

The intrinsic efficiency of a motor working under the above conditions is

$$\frac{E'}{E}.$$

Thus we see that to have a great **intrinsic** efficiency the motor should be allowed to run at a great speed, and develop a large back E. M. F.

The part of the work lost in heating bearings, &c., generally increases with the speed, so that the **commercial** efficiency does not continually increase with the speed.

To find when the work done per second, or the **activity**, is greatest, let us put W for this quantity.

$$W = EI - RI^2.$$

For W to be a maximum for variations of I,

$$0 = E - 2RI,$$

$$I = \frac{E}{2R}.$$

Thus the back E. M. F. developed is $E/2$.

In this case the intrinsic efficiency is $1/2$.

13. Transmission of Power. Sometimes a dynamo is used at one station, where there is a source of energy, to develop a current and drive a motor at a distant station, where work is required to be done, by means of conducting wires between the two stations. This operation is generally called the 'transmission of power.'

Suppose we have a series dynamo and series motor.

Let E be the E. M. F. developed by the dynamo.

E' the back E. M. F. developed by the motor.

R the entire resistance of the circuit formed of dynamos, motor, and conductors.

I the current passing.

The electrical energy developed per second is

$$EI = \frac{E(E-E')}{R}.$$

The work done per second by the motor is

$$E'I = \frac{E'(E-E')}{R}.$$

The work lost in heating conductors is

$$RI^2 = \frac{(E-E')^2}{R}.$$

The intrinsic efficiency of the arrangement is

$$\frac{E'}{E}.$$

Suppose we increase E and E', keeping $E-E'$ constant. Then the energy lost in heating conductors remains the same as before; but both the efficiency E'/E, and the activity $E'(E-E')/R$, are increased.

Thus it is advantageous to run at high E. M. F.s.

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THE END.

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